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SEARCH FOR MOVING TARGETS

PACIFIC-SIERRA RESEARCH CORPORATION
SANTA MONICA, CALIFORNIA

NOVEMBER 1976

PSR Report 619

Search for Moving Targets

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November 1976

Final Report for Period 1 April 1976 - 30 September 1976

Prepared for

Office of Naval Research
800 North Quincy Street
Arlington, Virginia 22217

Available for Public Release; Distribution Unlimited

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1456 Cloverfield Blvd. Santa Monica, California 90404

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)

"SEARCH FOR MOVING TARGETS"

5. TYPE OF REPORT & PERIOD COVERED

Final Report
1 Apr - 30 Sept 1976

6. PERFORMING ORG. REPORT NUMBER

PSR-619

7. AUTHOR(s)

Anthony P. Ciervo

8. CONTRACT OR GRANT NUMBER(s)

N00014-76-C-0792

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Pacific-Sierra Research Corporation
1456 Cloverfield Boulevard
Santa Monica, California 90404

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS

Office of Naval Research
800 N. Quincy Street
Arlington, Virginia 22217

12. REPORT DATE

November 1976

13. NUMBER OF PAGES

75

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)

Unclassified

15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Available for Public Release; Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

search theory, probability, Markov processes, diffusion processes, detection, sonobuoy, ASW tactics

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A fundamental problem in the theory of search involves the calculation of the probability of detection for searchers following known paths while attempting to detect a target whose motion is characterized statistically. The searchers' laws of detection and the target's initial distribution are given. Hellman has solved the problem when provided a Fokker-Planck equation for target motion, an unlikely input

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

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for military applications. We solve the problem using the transition probabilities directly, and present a closed-form expression for the probability of detecting a fleeing datum with an arbitrary search density. This solution is used to perform a gradient optimization for placement of stationary searchers in an ASW context.

PREFACE

This report presents a new formulation in search theory inspired by Pacific-Sierra Research Corporation's continuing analysis of ASW problems. The formulation is distinguished by 1) a realistic characterization of the search environment relative to prior analytical models, and 2) low computational costs compared to simulation techniques. The report should be of interest to operations researchers responsible for for optimizing search tactics and assessing search technology.

ACKNOWLEDGMENTS

The following individuals have made valuable suggestions at various points in the development of the search formulation: J. L. Wyatt of Lawrence Livermore Laboratories; J. B. McQueen of the University of California at Los Angeles; and R. F. Lutomirski, K. V. Saunders, and M. A. Dore of Pacific-Sierra Research Corporation. J. G. Smith of the Office of Naval Research monitored the research.

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1.0 INTRODUCTION AND SUMMARY

A fundamental problem in search theory is to calculate the probability of detection for searchers attempting to find a target whose initial location and subsequent motion are characterized statistically. The searchers' paths and laws of detection are assumed known. The solution of this problem has important applications in the optimization of search tactics and the assessment of search technology. Because of a lack of relevant and accessible research on the moving target problem, operations analysts have often resorted to overly simple analytical procedures or expensive computer simulations to obtain numerical results.

This report presents a new formulation that provides an exact solution for the search problem, assuming a Markovian search and a target whose motion is a diffusion process. These assumptions are then relaxed to apply the search formulation to several important non-Markovian targets and searchers. In general our purpose is to 1) render the search formulation accessible to operations analysts without sacrificing mathematical rigor; 2) narrow the gap between certain analytical assumptions and the behavior of actual targets and existing search technology; and 3) illustrate the utility of the formulation with simple numerical examples. After a brief review of previous related research, the present section summarizes the developments in this report.

1.1 PRIOR RESEARCH

In his pioneering work [1], Bernard Koopman solves problems involving the search for moving targets with his classical "random search"

detection model. Although it remains a viable analytical tool thirty years after its introduction, the random search model is limited by omission of certain operational considerations such as a realistic representation of either target or searcher motion, or an accommodation of general detection laws.

Later results in search theory have generally emphasized the optimal allocation of search effort for detecting targets that are either stationary or have simple kinds of motion. The recent book by Stone [2] presents a unified treatment for much of this work.

Hellman [3] provides a general formulation of the Markovian search problem that is similar in many respects to developments given here. However, the level of presentation and the use of a Fokker-Planck equation to describe target motion may have limited the application of Hellman's work by those seeking computational results for actual search problems. For comparison, we describe the work of Koopman and Hellman in somewhat more detail after developing our formulation of the search problem.

1.2 SUMMARY

Section 2.0 considers the search for a Markovian target that moves discretely in both time and space. That is, the target may be situated in any one of a countably infinite number of locations, and is capable of jumping to another location at each time step. The initial distribution and the location-to-location transition probabilities for the target are known, as well as the probability that the searchers will find the target, given its location. The latter is determined by the

searchers' paths and laws of detection. We derive a recursive difference equation for the distribution of the target at any time conditioned on an unsuccessful search until that time. This expression is then used to obtain the probability of detection. Although it only crudely represents the movement of real targets, the discrete formulation is a good model for certain generalized search problems, as we show in an example involving the interception and decoding of clandestine messages.

The discrete search problem of Sec. 2.0 also introduces the continuous case treated in Sec. 3.0. Working similarly but at a somewhat higher mathematical level, that section derives a nonlinear integrodifferential equation for the target location density conditioned on an unsuccessful search. The target is assumed to move as a diffusion process with known initial density and transition probabilities. The searchers' ability to find the target is represented by the Markovian search density introduced by Koopman and used throughout the literature. By working with the joint events--target location and unsuccessful search--we linearize the integrodifferential equation and use its solution to establish a simple expression for the probability of detection. Section 3.0 concludes with a physical interpretation of the continuous search formulation, comparing it with the treatment by Hellman.

Section 4.0 applies the continuous search formulation to certain operational aspects of actual searches. The search density of Sec. 3.0 is expressed as an explicit function of the individual laws of detection and paths of an arbitrary number of moving or stationary searchers. Using Koopman's lateral range curve, we find an approximation that

extends the scope of the search formulation to many important non-Markovian search devices. The approximation involves the inversion of an Abel integral equation to obtain the "equivalent" Markovian law of detection from any well-behaved lateral range curve.

That section also extends the search formulation to include a simple but important class of non-Markovian targets of which the classical fleeing datum is a special case. The fleeing datum is a target that initially selects a heading at random in the interval 0° to 360°, and then move on a straight-line course at a known constant speed. Closed-form solutions of the differential equations derived in Sec. 3.0 are obtained for such a target. These solutions are used in a numerical example to illustrate the dependence of the search path on detection probability; the result is compared to one obtained with the path-invariant random search model of Koopman.

The last section concentrates on the numerical optimization of search effort in a problem dealing with stationary sensors used to detect a fleeing datum with an initial circular normal distribution. The sensors are assumed to have a definite-range law of detection. We use a simple and inexpensive computational procedure to find the optimal circular sensor pattern that maximizes the probability of detection for a search of fixed duration. The optimization is performed over a range of input parameters relevant to a submarine search. For a specific set of inputs, the best circular pattern with eight sensors is shown to be superior to an optimal square pattern using nine sensors.

Finally, we consider the effect of target course changes subsequent to the initial heading selection of the fleeing datum. The search is undertaken by sensors in a circular pattern previously optimized for a specific fleeing datum problem. All inputs remain as in that problem, except that the target chooses a new course every fixed time increment. The surprising result is that additional course selections hardly increase the target's chance of escape; in fact, they decrease its chance of escape if not sufficiently frequent. Section 5.0 concludes with suggestions for broadening our computational experience and extending the analytical development of the search formulation.

2.0 SEARCH IN DISCRETE SPACE AND TIME

This section considers the search for a Markovian target that moves in discrete time in a space consisting of a denumerable set of states or locations. The discrete search problem provides a structure in which we can consider the more complex situation where the search process and target movement occur in continuous time and space (i.e., where the target moves as a diffusion process). The continuous case, treated in Sec. 3.0, is the most relevant to real-world search problems, and, happily, will lead to closed-form solutions for the cumulative probability of detection in many problems of interest.

The discrete search problem, however, has intrinsic importance beyond merely introducing the continuous case as demonstrated by the example at the end of this section. In addition, search problems are often part of a larger model that has a previously defined discrete space for target motion. The discrete formulation is therefore especially useful in a large-scale simulation of a military operation where search is one of many functions that contribute to its overall success. Such simulations are handled almost exclusively by digital computers where friendly and enemy forces are constrained to move on a finite grid in discrete time steps.

Ignoring for the moment the distinctions between the discrete and continuous cases, we can make the following informal description of the general class of search problems addressed: Find an expression for the cumulative probability of detection for a given number of moving or stationary searchers attempting to detect a target whose location at

$t = 0$ and motion for $t \geq 0$ are governed by a specified initial distribution and transition probability function, respectively. Each searcher's path (fixed point for a stationary searcher) and law of detection are also given. Our approach is to first derive an expression for the location density of the target for any time $t \geq 0$, conditioned on an unsuccessful search in the interval $[0,t)$. The solution of this expression is then used to establish $P(t)$, the probability of detection in the interval $[0,t)$. For the discrete case, this procedure entails finding an expression for $P(n)$, the probability of detection by time n, after establishing a recursion for the distribution of the target at n given that the target has not been detected.

2.1 TARGET MOTION

Let the location of the target at time n be represented by $\{X_n, n = 0,1,2,\ldots\}$, an irreducible Markov process defined on the state space $\mathfrak{X}$ taken to be the set of positive integers $x = 1,2,3,\ldots$. Hence, target motion is considered to be a discrete Markov process with respect to both the state variable x and the time variable n.

At $n = 0$, target location is characterized by the known *initial target location density* $\rho_0(x)$, defined by

$$\rho_0(x) = \mathcal{P}\{X_0 = x\} \quad , \quad x \in \mathfrak{X} \quad ,$$

such that

$$\sum_{x=1}^{\infty} \rho_0(x) = 1 \quad .$$

Target movement is represented by the known stationary one-step *transition probabilities*

$$\psi_{ij} = \mathcal{P}\{X_{n+1} = j \mid X_n = i\} \quad ,$$

such that, for $i,j \in \mathfrak{X}$, and $n = 0,1,2,\ldots$, we have as usual

$$\text{(i)} \qquad \psi_{ij} \geq 0$$

and

$$\text{(ii)} \qquad \sum_{j=1}^{\infty} \psi_{ij} = 1 \quad .$$

n ordering target transition and search events, a prerequisite for the analysis to follow, we assume that the n^{th} target transition occurs instantaneously at time $n - \varepsilon$, $n = 1,2,\ldots$, where $\varepsilon << 1$ is arbitrarily small.

2.2 SEARCH PROCESS

Consider the detection event defined by

$$B_n : \{\text{target detection at time } n + \varepsilon\} \quad ,$$

where it is assumed that all $x \in \mathfrak{X}$ are searched instantaneously at each time $n + \varepsilon$, $n = 0,1,2,\ldots$, and (as in the case of target transitions) $\varepsilon << 1$ is arbitrarily small. The ability of the searcher to detect the target is characterized through the known *search density* $\gamma(x,n)$ defined by

$$\gamma(x,n) = \mathscr{P}\{B_n \mid X_n = x\}$$

for all $x \in \mathfrak{X}$ and $n = 0,1,2,\ldots$.

We remark that the search density $\gamma(x,n)$ results from the combined effect of a finite number of stationary or moving searchers, each with its own (possibly unique) detection rule and search path (a fixed point in $\mathfrak{X}$ for a stationary searcher). Furthermore, the search--more generally, each searcher--may commence at some time n' after the initial target "fix" with associated density $\rho_o(x)$; in this case, $\gamma(x,n) \equiv 0$ for all $x \in \mathfrak{X}$, and $n = 0,1,2,\ldots,n'-1$. Both of these points are elaborated in Sec. 4.0, which considers the applications for the continuous search formulation.

We limit the present analysis to a *Markovian search* in the sense that the result of a search at time $n + \varepsilon$ depends only on the location of the target at that time, and not the results of prior searches. More precisely,

$$\mathscr{P}\{B_n \mid X_n = i, \overline{B}_{n-1}, \overline{B}_{n-2}, \ldots, \overline{B}_o\}$$

$$= \mathscr{P}\{B_n \mid X_n = i\}$$

$$= \gamma(i,n) \quad .$$

The Markovian property has two interpretations. If we define the search as finished at the time of the first detection, then a search at time $n + \varepsilon$ implies the realization of the events $\overline{B}_{n-1}, \overline{B}_{n-2}, \ldots, \overline{B}_o$, and the property has a trivial meaning. On the other hand, if we choose

to continue the search indefinitely, then the Markovian property allows us to count detections without altering our search strategy (remembering that $\gamma(x,n)$ is fixed *a priori*). In the development below, we solve for the probability of the *first* detection by time n. Thus, both interpretations are viable.

2.3 INDEPENDENCE OF TARGET MOTION AND SEARCH

On intuitive grounds, the target is unaware of the search, and the searcher's strategy is fixed *a priori;* target motion and search should thus be independent in some sense. To embody this intuitive notion concretely, we introduce the *independence property:*

$$\mathscr{P}\{X_{n+1} = j \mid X_n = i, \bar{B}_n, \bar{B}_{n-1}, \ldots, \bar{B}_o\}$$

$$= \mathscr{P}\{X_{n+1} = j \mid X_n = i\}$$

$$= \psi_{ij} \quad .$$

This property simply states that the $(n+1)^{th}$ transition is independent of the previous n searches. Rewriting it in a slightly different form,

$$\mathscr{P}\{X_{n+1} = j, B_n \mid X_n = i, \bar{B}_{n-1}, \bar{B}_{n-2}, \ldots, \bar{B}_o\}$$

$$= \mathscr{P}\{X_{n+1} = j \mid X_n = i\} \mathscr{P}\{D_n \mid X_n = i\}$$

$$= \psi_{ij} \, \gamma(i,n) \quad ,$$

where we have employed the Markovian search property as well.

2.4 SEARCH THEOREMS

In order to establish $P(n)$, the probability of detection by time n, we introduce $\rho(x,n)$, the target location density at time n, given an unsuccessful search until that time--or, more simply, the *conditional density*. Thus,

$$\rho(x,n) = \mathscr{P}\{X_n = x \mid \overline{B}_{n-1}, \ldots, \overline{B}_o\} \tag{2.1}$$

for $x \in \mathfrak{X}$. The following theorem obtains a recursion for $\rho(x,n)$ from the givens $\rho_o(x)$, ψ_{ij}, and $\gamma(x,n)$.

THEOREM 2.1:

The conditional density $\rho(i,n)$ satisfies the recursive equation

$$\rho(j,n+1) = \frac{\sum_{i=1}^{\infty} [1-\gamma(i,n)]\ \psi_{ij}\ \rho(i,n)}{\sum_{i=1}^{\infty} [1-\gamma(i,n)]\ \rho(i,n)}, \quad n = 0, 1, 2, \ldots \tag{2.2}$$

with initial condition $\rho(i,o) = \rho_o(i)$, $i \in \mathfrak{X}$.

PROOF:

From Eq. (2.1) we have

$$\rho(j,n+1) = \mathscr{P}\{X_{n+1} = j \mid \overline{B}_n, \ldots, \overline{B}_o\}$$

$$= \frac{\mathscr{P}\{X_{n+1} = j,\ \overline{B}_n, \ldots, \overline{B}_o\}}{\mathscr{P}\{\overline{B}_n, \ldots, \overline{B}_o\}}$$

$$= \frac{\mathscr{P}\{X_{n+1} = j,\ \overline{B}_n \mid \overline{B}_{n-1}, \ldots, \overline{B}_o\}}{\mathscr{P}\{\overline{B}_n \mid \overline{B}_{n-1}, \ldots, \overline{B}_o\}} \quad ,$$

where

$$\mathscr{P}\{\overline{B}_n \mid \overline{B}_{n-1}, \ldots, \overline{B}_o\} = \sum_{i=1}^{\infty} \mathscr{P}\{X_n = i,\ \overline{B}_n \mid \overline{B}_{n-1}, \ldots, \overline{B}_o\}$$

$$= \sum_{i=1}^{\infty} \mathscr{P}\{\overline{B}_n \mid X_n = i,\ \overline{B}_{n-1}, \ldots, \overline{B}_o\} \mathscr{P}\{X_n = i \mid \overline{B}_{n-1}, \ldots, \overline{B}_o\}$$

$$= \sum_{i=1}^{\infty} \mathscr{P}\{\overline{B}_n \mid X_n = i,\ \overline{B}_{n-1}, \ldots, \overline{B}_o\}\, \rho(i,n) \quad .$$

From the Markovian search property,

$$\mathscr{P}\{\overline{B}_n \mid X_n = i,\ \overline{B}_{n-1}, \ldots, \overline{B}_o\} = \mathscr{P}\{\overline{B}_n \mid X_n = i\}$$

$$= 1 - \gamma(i,n) \quad ,$$

so that

$$\mathscr{P}\{\overline{B}_n \mid \overline{B}_{n-1}, \ldots, \overline{B}_o\} = \sum_{i=1}^{\infty} [1-\gamma(i,n)]\ \rho(i,n) \quad .$$

Similarly,

$$\mathscr{P}\{X_{n+1} = j,\ \bar{B}_n \mid \bar{B}_{n-1},\ldots,\bar{B}_o\}$$

$$= \sum_{i=1}^{\infty} \mathscr{P}\{X_n = i,\ X_{n+1} = j,\ \bar{B}_n \mid \bar{B}_{n-1},\ldots,\bar{B}_o\}$$

$$= \sum_{i=1}^{\infty} \mathscr{P}\{X_{n+1} = j,\ \bar{B}_n \mid X_n = i,\ \bar{B}_{n-1},\ldots,\bar{B}_o\} \mathscr{P}\{X_n = i \mid \bar{B}_{n-1},\ldots,\bar{B}_o\}$$

$$= \sum_{i=1}^{\infty} \mathscr{P}\{X_{n+1} = j,\ \bar{B}_n \mid X_n = i,\ \bar{B}_{n-1},\ldots,\bar{B}_o\}\, \rho(i,n) \quad .$$

From the independence property,

$$\mathscr{P}\{X_{n+1} = j,\ \bar{B}_n \mid X_n = i,\ \bar{B}_{n-1},\ldots,\bar{B}_o\}$$

$$= \mathscr{P}\{X_{n+1} = j,\ \bar{B}_n \mid X_n = i\}$$

$$= \mathscr{P}\{X_{n+1} = j \mid X_n = i\}\, \mathscr{P}\{\bar{B}_n \mid X_n = i\}$$

$$= \psi_{ij}\ [1-\gamma(i,n)] \quad .$$

Therefore,

$$\mathscr{P}\{X_{n+1} = j,\ \bar{B}_n \mid \bar{B}_{n-1},\ldots,\bar{B}_o\}$$

$$= \sum_{i=1}^{\infty} \psi_{ij}\ [1-\gamma(i,n)]\ \rho(i,n) \quad ,$$

which completes the proof.

Now we let Q(n) be the probability of an unsuccessful search until time n. Thus,

$$Q(n) = \mathscr{P}\{\bar{B}_{n-1}, \bar{B}_{n-2}, \ldots, \bar{B}_o\} \quad . \tag{2.3}$$

and the probability of detection by n is

$$P(n) = 1 - Q(n) \quad . \tag{2.4}$$

The following theorem establishes P(n) from knowledge of $\rho(i,n)$.

THEOREM 2.2:

The probability of detection by time n *is given by*

$$P(n) = 1 - \prod_{n'=0}^{n-1} \sum_{i=1}^{\infty} [1-\gamma(i,n')]\, \rho(i,n') \quad , \tag{2.5}$$

where $\gamma(i,o) = 0$ *and* $\rho(i,o) = \rho_o(i)$ *for all* $i \in \mathbf{X}$.

PROOF:

From Eq. (2.3) we have

$$\frac{Q(n+1)}{Q(n)} = \frac{\mathscr{P}\{\bar{B}_n, \ldots, \bar{B}_o\}}{\mathscr{P}\{\bar{B}_{n-1}, \ldots, \bar{B}_o\}}$$

$$= \mathscr{P}\{\bar{B}_n \mid \bar{B}_{n-1}, \ldots, \bar{B}_o\}$$

$$= \sum_{i=1}^{\infty} \mathscr{P}\{X_n = i,\ \overline{B}_n \mid \overline{B}_{n-1}, \ldots, \overline{B}_o\}$$

$$= \sum_{i=1}^{\infty} \mathscr{P}\{\overline{B}_n \mid X_n = i\} \mathscr{P}\{X_n = i \mid \overline{B}_{n-1}, \ldots, \overline{B}_o\}$$

$$= \sum_{i=1}^{\infty} [1-\gamma(i,n)]\ \rho(i,n) \quad .$$

Thus,

$$Q(n) = \prod_{n'=0}^{n-1} \sum_{i=1}^{\infty} [1-\gamma(i,n')]\ \rho(i,n') \quad ,$$

which together with Eq. (2.4) completes the proof.

2.5 DISCRETE SEARCH PROBLEM

As an application of the discrete search formulation, consider the following generalized search problem. Each day Red selects a broadcast channel to send an encoded message to remote operatives. In an attempt to intercept and decode Red's message, Blue also selects a channel each day. Both Red and Blue are limited to one selection daily, and we assume that commensurate sets of channels are available to each. We also assume that Blue has some knowledge of the process by which Red makes daily channel selections. The problem is to assess Blue's strategy for intercepting and decoding Red's message.

The "target" in this case is Red's broadcast channel. Let the Markov process $\{X_n, n = 1,2,\ldots\}$ with state space $\mathfrak{X}$ represent Red's daily channel selection. Thus, if $\mathfrak{X}$ is taken to be the finite set of integers $x = 1,2,\ldots,I$, then $X_n = x$ has the interpretation that Red chooses to broadcast on channel $x \in \mathfrak{X}$ for day n. We assume that, on day zero, all channels are equally likely, so $\rho_o(x) = I^{-1}$ for all $x \in \mathfrak{X}$. Blue's knowledge of Red's broadcast sequence is embodied by the one-step transition probabilities ψ_{ij} for all $i,j \in \mathfrak{X}$.

Blue's strategy, fixed *a priori*, consists of the monitored sequence of channels denoted by $\langle z_n \rangle$, such that $z_n \in \mathfrak{X}$ for $n = 0,1,2,\ldots$. If Blue has correctly selected Red's broadcast channel on any day, then he has probability γ_o of decoding Red's message. Blue's "search density" is thus given by

$$\gamma(x,n) = \begin{cases} \gamma_o & x = z_n \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in \mathfrak{X}$ and $n = 0,1,2,\ldots$. The probability that by day n Blue will have successfully decoded a message sent by Red is, from Eq. (2.5),

$$P(n) = 1 - \prod_{n'=0}^{n-1} \sum_{i=1}^{I} [1 - \gamma(i,n')]\rho(i,n') \qquad ,$$

where $\rho(i,n)$ is found from the recursive equation (2.2),

$$\rho(j,n+1) = \frac{\sum_{i=1}^{I} [1 - \gamma(i,n)]\psi_{ij}\, \rho(i,n)}{\sum_{i=1}^{I} [1 - \gamma(i,n)]\rho(i,n)} ,$$

with initial condition $\rho(i,0) = I^{-1}$ for all $i \in \mathfrak{X}$.

3.0 SEARCH FOR A TARGET WHOSE MOTION IS A DIFFUSION PROCESS

This section derives the key result of our analysis--a linear differential equation from which a simple expression for the probability of detection is obtained when the search is undertaken in continuous space and time. The target is assumed to move as a diffusion process with a known transition probability function and initial distribution. The search process is the continuous space and time analogue of the discrete Markovian search discussed in Sec. 2.0.

The importance of the continuous search problem is twofold. First, physical considerations suggest a continuous space and time representation of target and searcher behavior. Real targets generally cannot move over finite distances in zero time, and existing search devices generally operate continuously in time. Happily, these observations suggest that the processes used to define the search problem have little difficulty meeting the regularity and continuity conditions required for the derivations below.

Second, the continuous search formulation involves expressions that are easier to manipulate than the recursion of Theorem 2.1 for the discrete case. Indeed, Sec. 4.0 presents a closed-form solution for a classical real-world problem, the search for a fleeing datum. Only simple numerical integrations are required to compute the probability of detection for this problem.

The analysis of this section is necessarily carried out at a significantly higher mathematical level than that of Sec. 2.0 for the discrete search problem. However, the structure of the derivations remains the

same: establish a relation for the conditional target location density, and derive an expression for the probability of detection as a function of the conditional density. The integrodifferential equation for the conditional density is linearized by considering the joint density (i.e., the density associated with the joint event, target location, and unsuccessful search). Section 4.0 solves the linear integrodifferential equation for the joint density under the assumption of the classical fleeing-datum target motion and an arbitrary Markovian search.

3.1 TARGET MOTION

Consider the Markov process $\{X(t), t\geq 0\}$ with continuous time parameter $t\in[0,\infty)$ and state space $\mathfrak{X}$ taken to be E^2, the two-dimensional Euclidean space. The location of the target at time $t \geq 0$ is represented by the random variable $X(t)\in\mathfrak{X}$ with coordinates $(X_1(t), X_2(t))$, $-\infty < X_i(t) < \infty$.

At $t = 0$, the target is located according to the known initial density $\rho_o(x)$. Thus,

$$\rho_o(x)dx = \mathscr{P}\{X(0)\in dx\}$$

and

$$\int\rho_o(x)dx = 1 \qquad , \tag{3.1}$$

where dx is an infinitesimal element of area containing the point $x\in\mathfrak{X}$, and the integral is over E^2 (as will be the case for all integrals below, unless otherwise noted).

The known transition probabilities of $\{X(t), t\geq 0\}$ are given by the function $\psi(x,t;y,\tau)$, $\tau > t$, $x,y\in\mathfrak{X}$, such that

$$\psi(x,t;y,\tau)dy = \mathscr{P}\{X(\tau) \in dy \mid X(t) = x\} \tag{3.2}$$

and

$$\int \psi(x,t;y,\tau)dy = 1 \quad . \tag{3.3}$$

Since $\{X(t),\ t \geq 0\}$ is a Markov process, it satisfies the Chapman-Kolmogorov equation; namely, for $s \in (t,\tau)$,

$$\psi(x,t;y,\tau) = \int \psi(x,t;z,s)\ \psi(z,s;y,\tau)dz \quad . \tag{3.4}$$

We assume that $\psi(x,t;y,\tau)$ has a derivative with respect to τ at $\tau = t$, so that we may write for small Δt

$$\psi(x,t;y,t+\Delta t) = \delta(y-x) + \Delta t \left[\frac{\partial \psi(x,t;y,\tau)}{\partial \tau}\right]_{\tau=t} + o(\Delta t) \quad , \tag{3.5}$$

where $\delta(\cdot)$ denotes the two-dimensional Dirac delta function.

We will also need a set of regularity conditions that limit the behavior of the target over small intervals of time. These conditions will be recognized as those used in the derivation of the Kolmogorov diffusion equations. With S_δ taken to be a circle of radius $\delta > 0$ and center x, the regularity conditions for $x = (x_1, x_2)$, $x, y \in \mathfrak{X}$, $t \in [0,\infty)$, and small Δt are given by

$$\int_{E^2 - S_\delta} \psi(x,t;y,t+\Delta t)dy = o(\Delta t) \quad , \tag{3.6}$$

$$\int_{S_\delta} (y_i - x_i)\ \psi(x,t;y,t+\Delta t)dy = a_i(x,t)\Delta t + o(\Delta t) \quad , \quad i = 1,2 \quad , \tag{3.7}$$

$$\int_{S_\delta} (y_i-x_i)\ (y_j-x_j)\ \psi(x,t;y,t+\Delta t)dy$$

$$= b_{ij}(x,t)\Delta t + o(\Delta t) \quad , \qquad i,j = 1,2 \quad , \qquad (3.8)$$

and

$$\int_{S_\delta} (y_i-x_i)^u\ (y_j-x_j)^v\ \psi(x,t;y,t+\Delta t)dy = o(\Delta t) \quad , \qquad i,j = 1,2 \quad , \quad (3.9)$$

where, for Eq. (3.9), $u,v \geq 0$ and $u + v > 2$.

3.2 SEARCH PROCESS

The search is undertaken continuously in time and is Markovian in nature. If we denote the detection event for $\tau > t$ by

$$B(t,\tau): \{\text{target detection within } [t,\tau)\} \quad ,$$

then the known search density $\gamma(x,t)$ is defined by

$$\gamma(x,t)\Delta t + o(\Delta t) = \mathscr{P}\{B(t,t+\Delta t) \mid X = x\} \qquad (3.10)$$

for $x \in \mathfrak{X}$, $t \in [0,\infty)$, and small Δt. Note that the probability in Eq. (3.10) is conditioned on a *stationary* target at x. We will expend considerable effort to suit our definition of $\gamma(x,t)$ to the case of a target that, while moving, satisfies the regularity conditions (3.6) to (3.9). To this end, we require that, for all $x \in \mathfrak{X}$, $t \in [0,\infty)$, a constant M exist such that

$$\gamma(x,t) \leq M < \infty \quad ; \qquad (3.11)$$

and, for all $x,y \in \mathfrak{X}$, $t \in [0,\infty)$, and small Δt,

$$\mathscr{P}\{B(t,t+\Delta t) \mid X(t+\Delta t) = y,\ X(t) = x\}$$

$$= \left\{\gamma(x,t) + \sum_{n=1,2} 1/n! \left[d_1\partial/\partial x_1 + d_2\partial/\partial x_2\right]^n \gamma(x,t)\Big|_{x=y}\right.$$

$$\left. + o(|y-x|^2)\right\}\Delta t + o(\Delta t) \qquad , \qquad (3.12)$$

where we have made use of the notation

(i) $d_i = (y_i - x_i)$,

(ii) $[d_1\partial/\partial x_1 + d_2\partial/\partial x_2]^2$

$$= [d_1^2\partial^2/\partial x_1^2 + 2d_1d_2\partial^2/\partial x_1\partial x_2 + d_2^2\partial^2/\partial x_2^2] \quad ,$$

(iii) $o(|y-x|^2) = o(d_1^2) + o(d_1d_2) + o(d_2^2)$.

Furthermore, we assume that the first- and second-order spatial partials of $\gamma(x,t)$ exist and are bounded for all $x \in \mathbf{X}$ and $t \in [0,\infty)$.

Equation (3.12) bears further explanation. It embodies the notion th t, if we *know* that the target remains in the immediate neighborhood of x over a small interval of time Δt, then we can express the probability of detection in $[t,t+\Delta t)$ by an expansion of $\gamma(\cdot,t)\Delta t + o(\Delta t)$ about the point x. Indeed, Lemma 3.1 below proves that, since the target cannot leave the neighborhood of x in a small interval of time, it is unnecessary to condition on the nearby endpoint $X(t+\Delta t) = y$.

As in the discrete case, we must formalize the Markovian nature of the search. We thus require that the search process satisfy

$$\mathscr{P}\{B(t,\tau) \mid X(t)=x,\ \bar{B}(0,t)\} = \mathscr{P}\{B(t,\tau) \mid X(t)=x\} \tag{3.13}$$

and

$$\mathscr{P}\{B(t,t+\Delta t) \mid X(t+\Delta t)=y,\ \bar{B}(0,t)\} = \mathscr{P}\{B(t,t+\Delta t) \mid X(t+\Delta t)=y\} + o(\Delta t) \tag{3.14}$$

for all $x,y\in\mathfrak{X}$, $t\in[0,\infty)$, and small Δt. Finally, the independence of search and target motion is embodied by the obvious requirement that

$$\mathscr{P}\{X(\tau)\in dy \mid X(t)=x,\ \bar{B}(0,t)\} = \mathscr{P}\{X(\tau)\in dy \mid X(t)=x\} \tag{3.15}$$

for all $x,y\in\mathfrak{X}$, $\tau > t$, and $\tau,t\in[0,\infty)$.

3.3 CONDITIONAL DENSITY

The conditional density $\rho(x,t)$ is defined for dx, an infinitesimal element of area containing x, by

$$\rho(x,t)dx = \mathscr{P}\{X(t)\in dx \mid \bar{B}(0,t)\} \quad , \tag{3.16}$$

where $x\in\mathfrak{X}$ and $t\in[0,\infty)$. This subsection derives a nonlinear integro-differential equation for $\rho(x,t)$ in terms of the givens $\rho_o(x)$, $\psi(x,t;y,\tau)$, and $\gamma(x,t)$.

The following lemma extends the definition of $\gamma(x,t)$ to a moving target that satisfies the regularity conditions given by Eqs. (3.6) to (3.9).

LEMMA 3.1:

Given the process $\{X(t),\ t\geq 0\}$ *representing target location, the search density* $\gamma(x,t)$ *satisfies the following conditions for small* Δt:

$$A.\quad \mathscr{P}\{\overline{B}(t,t+\Delta t) \mid X(t)=x\} = 1-\gamma(x,t)\Delta t + o(\Delta t) \tag{3.17}$$

and

$$B.\quad \mathscr{P}\{\overline{B}(t,t+\Delta t) \mid X(t+\Delta t)=y\} = 1-\gamma(y,t)\Delta t + o(\Delta t) \quad . \tag{3.18}$$

PROOF:

For part A, we write

$$\mathscr{P}\{\overline{B}(t,t+\Delta t) \mid X(t)=x\}$$

$$= \int \mathscr{P}\{\overline{B}(t,t+\Delta t) \mid X(t+\Delta t)=y,\ X(t)=x\}\ \mathscr{P}\{X(t+\Delta t)\in dy \mid X(t)=x\} \quad ,$$

which by Eqs. (3.2), (3.3), and (3.12) becomes

$$\mathscr{P}\{\overline{B}(t,t+\Delta t) \mid X(t)=x\}$$

$$= 1-\gamma(x,t)\Delta t + \Delta t \sum_{n=1,2} 1/n! \int \left\{ \left[d_1 \partial/\partial x_1 + d_2 \partial/\partial x_2 \right]^n_{x=y} \gamma(x,t) + o(|y-x|^2) \right\}$$

$$\psi(x,t;y,t+\Delta t)dy + o(\Delta t) \quad .$$

The integral in this last expression need be performed only over S_δ, since the first- and second-order partials of $\gamma(x,t)$ are bounded and Eq. (3.6) applies. Furthermore, Eqs. (3.7) to (3.9) imply that performing the integral over S_δ will result only in Δt and $o(\Delta t)$ terms proving part A of the lemma. The proof of part B is similar, if we consider the process $\{X(t),\ t\geq 0\}$ reversed in time and impose similar regularity conditions.

We will need the following lemmas to obtain our main result in Theorem 3.1.

LEMMA 3.2:

$$\mathscr{P}\{\overline{B}(t,t+\Delta t) \mid \overline{B}(0,t)\} = 1 - \Delta t \int \gamma(x,t)\ \rho(x,t)dx + o(\Delta t) \quad . \tag{3.19}$$

PROOF:

Using Eqs. (3.13), (3.16), and (3.17), we quickly obtain

$$\begin{aligned}
&\mathscr{P}\{\overline{B}(t,t+\Delta t) \mid \overline{B}(0,t)\} \\
&\quad = \int \mathscr{P}\{\overline{B}(t,t+\Delta t) \mid X(t) = x\}\ \mathscr{P}\{X(t) \in dx \mid \overline{B}(0,t)\} \\
&\quad = \int [1 - \gamma(x,t)\Delta t + o(\Delta t)]\ \rho(x,t)dx \quad ,
\end{aligned}$$

which proves the lemma.

LEMMA 3.3:

For dy, *an infinitesimal element of area containing* $y \in \mathfrak{X}$ *and small* Δt,

$$\begin{aligned}
&\mathscr{P}\{X(t+\Delta t) \in dy \mid \overline{B}(0,t)\} \\
&\quad = dy \left\{ \rho(y,t) + \Delta t \int \left[\frac{\partial \psi(x,t;y,\tau)}{\partial \tau}\right]_{\tau=t} \rho(x,t)dx + o(\Delta t) \right\} \quad .
\end{aligned} \tag{3.20}$$

PROOF:

With help from Eqs. (3.2), (3.5), (3.15), and (3.16), we obtain

$$\begin{aligned}
&\mathscr{P}\{X(t+\Delta t) \in dy \mid \overline{B}(0,t)\} \\
&\quad = \int_x \mathscr{P}\{X(t+\Delta t) \in dy \mid X(t) = x\}\ \mathscr{P}\{X(t) \in dx \mid \overline{B}(0,t)\} \\
&\quad = dy \int \psi(x,t;y,t+\Delta t)\ \rho(x,t)dx
\end{aligned}$$

$$= dy \int \left[\delta(y-x) + \Delta t \left. \frac{\partial \psi(x,t;y,\tau)}{\partial \tau} \right|_{\tau=t} + o(\Delta t) \right] \rho(x,t)dx$$

$$= dy \left\{ \rho(y,t) + \Delta t \int \left[\frac{\partial \psi(x,t;y,\tau)}{\partial \tau} \right]_{\tau=t} \rho(x,t)dx + o(\Delta t) \right\} \quad ,$$

...ving the lemma.

For notational convenience, let

$$\Gamma(t) = \int \gamma(x,t)\ \rho(x,t)dx \tag{3.21}$$

for all $t \in [0,\infty)$. The function $\Gamma(t)$ will be recognized as the *search int... ty* in the sense that the probability of detection in the interval $[t, t\ \ )$, conditioned on an unsuccessful search in $[0,t)$, is $\Gamma(t)\Delta t + o(\Delta t)$. Indeed, Lemma 3.2 tells us that

$$\mathscr{P}\{B(t,t+\Delta t) \mid \overline{B}(0,t)\} = \Gamma(t)\Delta t + o(\Delta t) \quad . \tag{3.22}$$

The following theorem presents an integrodifferential equation for the conditional density $\rho(x,t)$.

THEOREM 3.1:

Given the inputs $\rho_o(x)$, $\psi(x,t;y,\tau)$, *and* $\gamma(x,t)$ *for the continuous search problem, the conditional target location density* $\rho(y,t)$, $y \in \mathfrak{X}$, $t \in [0,\infty)$ *s the solution of the nonlinear integrodifferential equation*

$$\frac{\partial \rho(y,t)}{\partial t} = \int \left[\frac{\partial \psi(x,t;y,\tau)}{\partial \tau} \right]_{\tau=t} \rho(x,t)dx + \rho(y,t)\ [\Gamma(t) - \gamma(y,t)] \tag{3.23}$$

with initial condition $\rho(y,0) = \rho_o(y)$ *for all* $y \in \mathfrak{X}$.

PROOF:

From Eq. (3.16), we have

$$\rho(y,t+\Delta t)dy = \mathcal{P}\{X(t+\Delta t)\in dy \mid \bar{B}(t,t+\Delta t),\ \bar{B}(0,t)\}$$

$$= \frac{\mathcal{P}\{X(t+\Delta t)\in dy,\ \bar{B}(t,t+\Delta t) \mid \bar{B}(0,t)\}}{\{\bar{B}(t,t+\Delta t) \mid \bar{B}(0,t)\}}$$

$$= \frac{\mathcal{P}\{X(t+\Delta t)\in dy \mid \bar{B}(0,t)\}\ \mathcal{P}\{\bar{B}(t,t+\Delta t) \mid X(t+\Delta t)=y,\ \bar{B}(0,t)\}}{\mathcal{P}\{\bar{B}(t,t+\Delta t) \mid \bar{B}(0,t)\}} \quad .$$

Noting that, for small ε,

$$(1-\varepsilon)^{-1} = 1 + \varepsilon + 0(\varepsilon) \qquad ,$$

with the help of Eq. (3.22) we can write

$$\left[\mathcal{P}\{\bar{B}(t,t+\Delta t) \mid \bar{B}(0,t)\right]^{-1}$$

$$= 1 + \Gamma(t)\Delta t + o(\Delta t) \qquad .$$

Together with Eq. (3.14) and Lemmas 3.1 and 3.3, this expression results in

$$\rho(y,t+\Delta t) = \Bigg\{\rho(y,t) + \Delta t\int\left[\frac{\partial\psi(x,t;y,\tau)}{\partial\tau}\right]_{\tau=t}\rho(x,t)dx$$

$$+ o(\Delta t)\Bigg\}\ [1 - \gamma(y,t)\Delta t + o(\Delta t)]\ [1 + \Gamma(t) + o(\Delta t)]$$

$$= \rho(y,t) + \Delta t\ \Bigg\{\int\left[\frac{\partial\psi(x,t;y,t)}{\partial\tau}\right]_{\tau=t}\rho(x,t)dx$$

$$+ \rho(y,t)\ [\Gamma(t) - \gamma(y,t)]\Bigg\} + o(\Delta t) \qquad .$$

Transposing, dividing by Δt, and letting $\Delta t \to 0$ completes the proof.

Corollary 3.1 shows that the solution of Eq. (3.23) is indeed a proper density for all $t \in [0,\infty)$.

COROLLARY 3.1:

The solution of the equation for the conditional density presented in Theorem 3.1 satisfies $\int \rho(y,t)dy = 1$ *for all* $t \in [0,\infty)$.

PROOF:

Let $w(t) = \int \rho(y,t)dy$ for all $t \in [0,\infty)$. Then Eq. (3.23) becomes

$$\partial/\partial t \; w(t) = u(t) + v(t) \quad ,$$

where

$$u(t) = \int_y \int_x \left[\frac{\partial \psi(x,t;y,\tau)}{\partial \tau}\right]_{\tau=t} \rho(x,t)dxdy$$

$$= \int_x \rho(x,t) \left[\partial/\partial\tau \int_y \psi(x,t;y,\tau)dy\right]_{\tau=t} dx$$

and

$$v(t) = \int \rho(y,t) \; [\Gamma(t) - \gamma(y,t)]dy$$

$$= \Gamma(t) \; [w(t) - 1] \quad .$$

By Eq. (3.3), however, $u(t) = 0$ for $t \in [0,\infty)$, so that

$$\partial/\partial t \; w(t) = \Gamma(t) \; [w(t) - 1] \quad .$$

The solution of this equation is

$$w(t) = 1 + K\, e^{\int_0^t \Gamma(\xi)d\xi} \quad .$$

But $w(o) = 1$ by Eq. (3.1), completing the proof.

We are now prepared to consider the probability of detection $P(t)$. Specifically,

$$Q(t) = \mathscr{P}\{\overline{B}(0,t)\} \tag{3.24}$$

and

$$P(t) = 1 - Q(t) \tag{3.25}$$

for all $t \in [0,\infty)$. The following theorem obtains $P(t)$ from the search intensity $\Gamma(t)$.

THEOREM 3.2:

The probability of finding a target by time t *when the target moves as a diffusion process is given by*

$$P(t) = 1 - \exp[-\int_0^t \Gamma(\xi)d\xi] \quad , \tag{3.26}$$

where the search intensity, $\Gamma(t) = \int \gamma(x,t)\, \rho(x,t)dx$, *is expressed in terms of* $\rho(x,t)$, *the solution to the equation of Theorem 3.1.*

PROOF:

From Eqs. (3.22) and (3.24),

$$\begin{aligned} Q(t+\Delta t) &= \mathscr{P}\{\overline{B}(0,t+\Delta t)\} \\ &= \mathscr{P}\{\overline{B}(0,t)\}\ \mathscr{P}\{\overline{B}(t,t+\Delta t)\ |\ \overline{B}(0,t)\} \\ &= Q(t)\ [1 - \Gamma(t)\Delta t + o(\Delta t)] \quad . \end{aligned}$$

Thus,

$$\frac{Q(t+\Delta t) - Q(t)}{Q(t)\Delta t} = -\Gamma(t) + \frac{o(\Delta t)}{\Delta t} \quad ,$$

and, after letting $\Delta t \to 0$, and noting that $Q(0) = 0$, we obtain

$$\log Q(t) = -\int_0^t \Gamma(\xi)d\xi \quad ,$$

proving the theorem.

3.4 LINEAR SEARCH EQUATION

The nonlinearity of Eq. (3.23) makes closed-form solutions for the conditional density $\rho(x,t)$ and probability of detection $P(t)$ difficult to come by. By considering the *joint* target location density (or, more simply, the joint density) $p(x,t)$, we can obtain a linear search equation and a simpler expression for $P(t)$. Thus, we define for all $x \in \mathcal{X}$ and $t \in [0,\infty)$

$$p(x,t)dx = \mathcal{P}\{X(t) \in dx, \overline{B}(0,t)\} \quad , \tag{3.27}$$

and quickly note that

$$p(x,t) = \rho(x,t)\,[1 - P(t)] \tag{3.28}$$

and

$$P(t) = 1 - \int p(x,t)dx \quad . \tag{3.29}$$

THEOREM 3.3:

The joint target location density $p(y,t)$, $y \in \mathbf{X}$, $t \in [0,\infty)$ *satisfies the linear integrodifferential equation*

$$\frac{\partial p(y,t)}{\partial t} = \int \left| \frac{\partial \psi(x,t;y,\tau)}{\partial \tau} \right|_{\tau=t} p(x,t)dx - \gamma(y,t)\ p(y,t) \quad , \quad (3.30)$$

with initial condition $p(y,0) = \rho_o(y)$ *for all* $y \in \mathbf{X}$.

PROOF:

The proof is to simply transform Eq. (3.23) by Eq. (3.28).

Although the joint density has less intuitive appeal than the conditional density, it clearly represents the superior analytical tool for solving search problems. Equations (3.29) and (3.30) will thus be the operative equations as we apply the results of this section to the numerical problems of Secs. 4.0 and 5.0.

3.5 PHYSICAL INTERPRETATION OF THE SEARCH FORMULATION

The nonlinear equation for the conditional density, Eq. (3.23), and the linear equation for the joint density, Eq. (3.30), have simple but important physical interpretations. Both equations consist of two terms-- the diffusion term resulting in density changes due only to target movement, and the search term causing density changes due only to the effect of search. For example, consider the search for a stationary target, i.e., a target with the trivial transition probability function $\psi(x,t;y,\tau) = \delta(y-x)$. In this case, Eq. (3.23) becomes

$$\frac{\partial \rho(x,t)}{\partial t} = \rho(x,t)\ [\Gamma(t) - \gamma(x,t)] \quad , \qquad (3.31)$$

and $\rho(x,t)$ is affected only by the search density $\gamma(x,t)$. Alternatively, with no search--i.e., if $\gamma(x,t) = 0$ for all $x \in \mathbf{X}$ and $t \in [0,\infty)$--we obtain the search-free equation

$$\frac{\partial\rho(y,t)}{t} = \int\left[\frac{\partial\psi(x,t;y,\tau)}{\partial\tau}\right]_{\tau=t}\rho(x,t)dx \quad , \tag{3.32}$$

where $\rho(x,t)$ is affected only by the motion of the target. Similar observations can be made for the joint density, $p(x,t)$.

The situation is illustrated in Fig. 3.1, where it is assumed that the initial density $\rho_0(x)$ was circular normal, and that at $t = 0$ the target chose a heading from a uniform distribution on $[0,2\pi)$ and fled at a known speed (the classical fleeing datum). The (x_1,x_2) plane is the search region and the x_3 axis plots $\rho(x,t)$ for $t = T > T_1 > 0$, where the search by a single moving searcher commenced at T_1. Thus, $\rho(x,t)$ is the solution to Eq. (3.32) for $t \in [0,T_1)$, and Eq. (3.23) for $t \in [T_1,T)$. The details of the solution are taken up in Sec. 4.0; but for now we note that the wide depression in the center of the $\rho(x,T)$ surface is caused by the Bessel function spreading of a fleeing datum density, as first described by Koopman [1], and the narrow depression along the ridge is due to an unsuccessful search following this path.

Intuitively, we expect that if we search an area unsuccessfully, then the object of our search is less likely to be located in that area. This phenomenon caused the depression along the ridge of $\rho(x,T)$ in Fig. 3.1. Analytically, Eq. (3.23) tells us that, for points under the influence of the searcher (i.e., those for which $\gamma(x,t)$ dominates its average

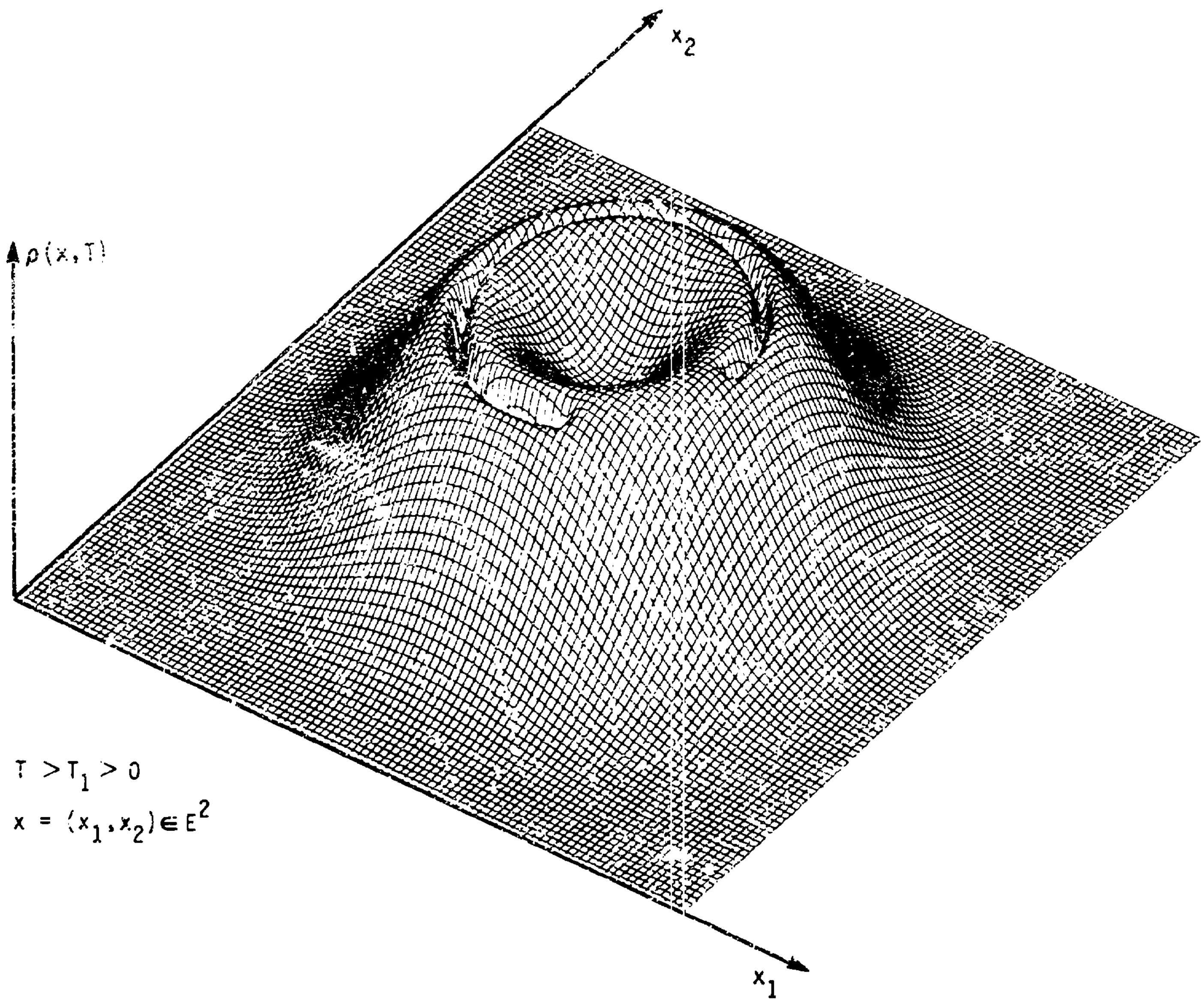


Fig. 3.1--Effect of target motion and search on an initial circular normal target location density

$\Gamma(t) = \int \gamma(x,t)\ \rho(x,t)dx$), the change of $\rho(x,t)$ with time due to the search is negative. The effect of $\Gamma(t)$ in Eq. (3.23) is to elevate $\rho(x,t)$ for points relatively free of the influence of the searcher, resulting in $\int \rho(x,t)dx = 1$ for all $t \in [0,\infty)$ as shown in Corollary 3.1. (In the linear Eq. (3.30), the search term $[-\gamma(x,t)\ p(x,t)]$ is always nonpositive so that $\int p(x,t)dx = 1 - P(t) \leq 1$ for all $t \in [0,\infty)$.)

The search-free equation, Eq. (3.32), is a direct result of the Chapman-Kolmogorov equation, Eq. (3.4). Indeed, we can write Eq. (3.4) for $t \in [0,\tau)$ as

$$\psi(x,0;y,\tau) = \int \psi(z,t;y,\tau)\ \psi(x,0;z,t)dz \quad .$$

Integrating over the initial density $\rho_o(x)$ results in

$$\int \psi(x,0;y,\tau)\ \rho_o(x)dx = \int \psi(z,t;y,\tau) \left[\int \psi(x,0,z,t)\ \rho_o(x)dx\right]dz \quad ;$$

or, in the absence of search,

$$\rho(y,\tau) = \int \psi(z,t;y,\tau)\ \rho(z,t)dz \quad ,$$

so that

$$\left.\frac{\partial \rho(y,\tau)}{\partial \tau}\right|_{\tau=t} = \int \left[\frac{\partial \psi(z,t;y,\tau)}{\partial \tau}\right]_{\tau=t} \rho(z,t)dz \quad ,$$

which is the search-free equation (3.32).

Finally, we remark that the only difference between Eq. (3.23) and the equation presented by Hellman [3] is in the diffusion term, because Hellman starts with a Fokker-Planck equation for $\psi(x,t;y,\tau)$, rather than

$\psi(x,t;y,\tau)$ itself. (Thus, by a manipulation similar to the one above for the Chapman-Kolmogorov equation, one could obtain a Hellman search-free equation from the Fokker-Planck equation.) The specification of the motion of an actual target by those responsible for the search would, needless to say, lead directly to a representative transition probability function, rather than a Fokker-Planck equation.

4.0 APPLICATIONS OF THE SEARCH FORMULATION

This section applies the results of Sec. 3.0 to certain operational aspects of actual searches. The emphasis is on target motions and search processes that ostensibly violate the Markovian assumptions needed to derive Eqs. (3.23) and (3.30), on which th continuous search formulation is based. This section should be regarded as an introduction to the work necessary to bridge the gap between the analytical assumptions made for tractability, and the behavior of actual targets and existing search devices.

Subsection 4.1 details the straightforward generalization of the search formulation to accommodate an arbitrary number of stationary or moving searchers. Recognizing that search devices are often not Markovian in the sense of Eq. (3.13), we present a good "equivalent" Markovian search density in subsection 4.2 by inverting the integral equation for Koopman's lateral range curve. Subsection 4.3 extends the search formulation to handle the simplest kind of non-Markovian target motion for which the classical fleeing datum is a special case. Closed-form solutions of the search equations are obtained for a fleeing datum search. Finally, to illustrate the generality of the search formulation, subsection 4.4 presents a numerical example in which the probability of detection is calculated as a function of time for two paths followed by a moving searcher attempting to find a fleeing datum; this result is compared to one obtained with a descendent of the path-invariant "random search" model first proposed by Koopman.

4.1 MULTIPLE SEARCHERS

The search density $\gamma(x,t)$ defined by Eq. (3.10) for a stationary target is actually the aggregate effect of the total search effort. The precise manner in which $\gamma(x,t)$ depends on the activities of many individual searchers does not affect the developments of Sec. 3.0, but is central to the application of the continuous search formulation. Thus, below we construct $\gamma(x,t)$ from the paths and laws of detection for each of an arbitrary number of moving or stationary searchers.

Consider a search undertaken by n searchers such that the i^{th} searcher follows the known path $z_i(t) \in \mathfrak{X}$ for $t \in [T_{i1}, T_{i2})$, where $T_{i1} \geq 0$ is the time at which the i^{th} searcher commences his effort, and $T_{i2} \in (T_{i1}, \infty)$ is the time at which he abandons the search. Obviously, for a stationary searcher, $z_i(t) = z_i \in \mathfrak{X}$ for $t \in [T_{i1}, T_{i2})$. The law of detection for the i^{th} searcher is defined in the same manner as $\gamma(x,t)$; i.e., the probability that the i^{th} searcher will detect in the small interval $[t, t+\Delta t)$, given that the target is at $x \in \mathfrak{X}$, is

$$\gamma_i[x, z_i(t)]\Delta t + o(\Delta t) \quad .$$

The n searchers are assumed independent in the sense that, from Eq. (3.10),

$$\gamma(x,t)\Delta t + o(\Delta t) = 1 - \prod_{i=1}^{n} \{1 - \gamma_i[x, z_i(t)]\Delta t + o(\Delta t)\}$$

$$= \Delta t \sum_{i=1}^{n} \gamma_i[x, z_i(t)] + o(\Delta t) \quad .$$

The search density $\gamma(x,t)$ is thus expressed in terms of the individual laws of detection $\gamma_i[x,z_i(t)]$, $i = 1,2,\ldots n$, by

$$\gamma(x,t) = \sum_{i=1}^{n} \gamma_i[x,x_i(t)] \quad . \tag{4.1}$$

Clearly, in order that the search process continue to satisfy Eqs. (3.11) through (3.15), the laws of detection for the individual searchers must satisfy similar conditions.

Noting that, for all $x \in \mathbf{X}$, $\gamma_i[x,z_i(t)] = 0$ when $t < T_{i1}$ or $t \geq T_{i2}$, we can define $T_2 - T_1$ as the *search duration* where

$$T_1 = \min_{1 \leq i \leq n} T_{i1}$$

and

$$T_2 = \max_{1 \leq i \leq n} T_{i2} \quad .$$

Obviously, $T_2 > T_1$ and $\gamma(x,t) = 0$ for $t < T_1$ or $t \geq T_2$. T_1 is commonly referred to as the "time late" in military applications.

4.2 NON-MARKOVIAN SEARCH: INVERSION OF A LATERAL RANGE CURVE

A sensor commonly used to search for submarines operates approximately as follows [4]: A signal that may indicate the presence of a submarine is integrated over a "sliding" time interval of fixed length. If the value of the integral ever exceeds a predetermined threshold, then the sensor is activated. Such a sensor is not Markovian in the sense of subsection 3.2 because of the "memory" property of the integral

operator. The present subsection develops a useful approximation for non-Markovian search devices that allows us to apply the search formulation to many situations where such devices are employed.

Whether or not a search device is Markovian, it can be characterized by the so-called *lateral range curve*, $q(\xi)$. The function $q(\xi)$ is defined as the probability of detecting a stationary target when the searcher follows a straight-line path of infinite length and closest approach ξ to the target [1]. The lateral range curve can be approximated by empirical data or computed directly from physical arguments.

Consider a search employing a single Markovian sensor with law of detection $\gamma_1[x, z_1(t)]$, which is a function of only the target-to-searcher Euclidean distance $r = |x - z_1(t)|$. If the target is stationary at the origin of E^2, and the searcher is stationary at $z_1 = (\xi, \mu) \in E^2$, then denote the search density by $\gamma(r) = \gamma_1[0, z_1]$. Now let the searcher move at speed w along the path from $(\xi, -\infty)$ to (ξ, ∞), thus obtaining the probability of detection $q(\xi)$ from the well-known expression [1]

$$q(\xi) = 1 - \exp\left[-F(\xi)\right] \quad , \tag{4.2}$$

where

$$F(\xi) = \frac{1}{w} \int_{-\infty}^{\infty} \gamma\left(\sqrt{\xi^2 + \mu^2}\right) d\mu \quad . \tag{4.3}$$

(Note that Eqs. (4.2) and (4.3) can also be obtained by use of Eqs. (3.29) and (3.30) for $\psi(x,t;y,\tau) = \delta(x)$ and $\rho(y,o) = \delta(y)$.) Clearly,

$q(\xi)$ is the lateral range curve for a Markovian device with law of detection $\gamma_1[x.z_1(t)]$.

The following question is reasonable to ask at this point: If we know the lateral range curve for a radially symmetric non-Markovian device, can we calculate the "equivalent" Markovian law of detection by inversion of Eq. (4.3)? As shown below, we can generally invert the lateral ra ge curve and then use the resulting Markovian law of detection as input to the search formulation.

In order to invert Eq. (4.3), write

$$F(\xi) = \frac{2}{w}\int_0^\infty \gamma\left(\sqrt{\xi^2 + \mu^2}\right)d\mu$$

$$= \frac{2}{w}\int_\xi^\infty \frac{\gamma(r)rdr}{(r^2 - \xi^2)^{1/2}} \quad ,$$

where, as before, $r = (\xi^2 + \mu^2)^{1/2}$. Now let $r^2 = 1/s$ and $\xi^2 = 1/a$, so that

$$F(a^{-1/2}) = \frac{a^{1/2}}{w}\int_0^a \frac{\gamma(s^{-1/2})ds}{s^{3/2}(a-s)^{1/2}}$$

or

$$G(a) = \int_0^a \frac{\lambda(s)ds}{(a-s)^{1/2}} \quad , \tag{4.4}$$

where

$$G(a) = w\, a^{-1/2}\, F(a^{-1/2})$$

and

$$\lambda(s) = s^{-3/2}\,\gamma(s^{-1/2}) \quad .$$

Equation (4.4) is an Abel integral equation [5], the solution of which, for continuously differentiable G(a), is

$$\lambda(s) = \frac{G(0)}{\pi s^{1/2}} + \frac{1}{\pi}\int_0^s \frac{G'(a)da}{(s-a)^{1/2}} \quad . \tag{4.5}$$

As an example of the use of Eq. (4.5), consider the lateral range curve for the inverse cube law of detection [1]

$$q(\xi) = 1 - \exp\left[-\frac{2k}{w\xi^2}\right] \quad ,$$

where w is searcher speed and k is a constant dependent on several aspects of the search that need not concern us now. Thus,

$$F(\xi) = \frac{2k}{w\xi^2} \quad ,$$

$$G(a) = 2k\,a^{1/2} \quad ,$$

and, from Eq. (4.5),

$$\lambda(s) = \frac{k}{\pi}\int_0^s \frac{da}{a^{1/2}(s-a)^{1/2}}$$

$$= \frac{2k}{\pi}\int_0^{s^{1/2}} \frac{du}{(s-u^2)^{1/2}}$$

$$= \frac{2k}{\pi} \left[\sin^{-1}\left(\frac{u}{s^{1/2}}\right) \right]_0^{s^{1/2}}$$

$$= k \quad .$$

Therefore, as expected $\gamma(r) = k/r^3$, the search density for the inverse cube law of detection.

Although we have shown how to obtain the equivalent law of detection for a non-Markovian search device, we have not shown how good an approximation it is for calculating the probability of detection using the search formulation of Sec. 3.0. Obviously, it is an exact approximation for infinitely long straight-line search paths, wherein lies the key to a general evaluation of the inversion technique. If the motion of the searcher *relative* to the target is nearly linear at almost constant speed over the detection range of the device, then the inversion technique represents a good approximation. Further analysis is necessary to establish the accuracy of the technique under more general search conditions.

4.3 CONDITIONALLY MARKOVIAN TARGETS: THE FLEEING DATUM

The search formulation is applied here to a member of the simple class of non-Markovian target motion described as follows: At $t = 0$, a realization of an n-dimensional random variable α is obtained from the known density $f(\alpha)$. Target motion for $t > 0$ is Markovian with conditional transition probability function $\psi_\alpha(x,t;y,\tau)$. Let the solution of Eq. (3.30) for $\psi_\alpha(x,t;y,\tau)$ be denoted by $p_\alpha(x,t)$. The joint density is then given by the n-dimensional integral $p(x,t) = \int p_\alpha(x,t)f(\alpha)d\alpha$, and the probability of detection remains $P(t) = 1 - \int p(x,t)dx$. This

class of motion, which we shall call conditionally Markovian, is a generalization of Stone's conditionally deterministic motion [2], where knowledge of α at $t = 0$ implies deterministic motion for $t > 0$. The fleeing datum, which we now treat in detail, is an important example of conditionally deterministic target motion.

The fleeing datum moves at a known constant speed u on a straight-line path after chooding a heading θ at $t = 0$ from a uniform distribution on $[0,2\pi)$. If θ is measured counterclockwise with respect to the x_1 coordinate, then the conditional transition probability function for the fleeing datum is given by the two-dimensional Dirac delta function

$$\psi_\theta(x,t;y,\tau) = \delta[(y-x) - v_\theta(\tau - t)] \quad , \tag{4.6}$$

where $v_\theta = u(\cos\theta, \sin\theta)$ is the random velocity vector with magnitude u and direction θ. Substituting Eq. (4.6) into Eq. (3.30) results in the following linear differential equation for the joint density conditioned on the bearing $\theta \in [0,2\pi)$:

$$\frac{\partial p_\theta(x,t)}{\partial t} = v_\theta \cdot \nabla\, p_\theta(x,t) - \gamma(x,t)\; p_\theta(x,t) \quad . \tag{4.7}$$

The solution of Eq. (4.7) for initial condition $p_\theta(x,0) = \rho_0(\ )$ is

$$p_\theta(x,t) = \rho_0(x - v_\theta t)\left[\exp -\int_0^t \gamma[x - v_\theta(t-s), s]ds\right] \tag{4.8}$$

for $x \in E^2$ and $t \in [0,\infty)$. Thus, the probability of detecting a fleeing datum by time $t \in [0,\infty)$ for initial location density $\rho_0(x)$ and search

density $\gamma(x,t)$ is given by

$$P(t) = 1 - \frac{1}{2\pi} \int_x \int_{\theta=0}^{2\pi} \rho_0(x - v_\theta t)\ \exp\left\{-\int_0^t \gamma[x - v_\theta(t-s), s]ds\right\} d\theta dx \quad , (4.9)$$

where $v_\theta = u(\cos\theta, \sin\theta)$ is the randomly oriented velocity vector with known magnitude u. This expression is used extensively for the numerical optimization procedure of Sec. 5.0.

4.4 COMPARISON WITH A RANDOM SEARCH MODEL

Prior to the work of Hellman [3] and the formulation presented here, problems concerning the search for a moving target were often treated by use of Monte Carlo simulations or the random search model of Koopman [1]. Simulations involve relatively high program development and execution costs, whereas the random search model omits many important features of the operational search problem, such as the searcher's actual path and realizable target motion. After a brief discussion of the random search model, this subsection solves a simple fleeing-datum search problem using that model and compares the result to the solution obtained using Eq. (4.9).

The random search model is predicated on the following assumptions:

1. The target's position is uniformly distributed in a region of area A and maintains that distribution for all $t \in [0,\infty)$.
2. The search path is random in A in the sense that disjoint sections of the path are distributed uniformly and independently in A.

3. The searcher's law of detection is such that detection is certain within range d of the target and cannot occur outside d.

Assumptions 1 and 2 do not correspond well to physical imperatives but it is asserted [1] that, in situations where target and searcher are moving in complex paths at varying speeds, the assumptions are reasonable. Assumption 3 is the well-known definite-range law of detection, which is often used as an approximation for a device with lateral range curve $q(\xi)$. In this case, d is taken to be $\int_0^\infty q(\xi)d\xi$.

Assumptions 1 through 3 lead to the well-known random search formula for the probability of detection

$$P = 1 - \exp\left(-\frac{2dL}{A}\right) ,$$

where L is the length of the searcher's path in A. If we now consider the search area to be a circle with increasing radius $R(\tau) = R_1 + u\tau$ (i.e., the radius is increasing at the rate at which a fleeing datum is assumed to be moving outward), then the probability of detection may be approximated by the random search formula for an expanding area [6]

$$P(\tau) = 1 - \exp\left[-\frac{2w\tau d}{\pi R_1(R_1 + u\tau)}\right] , \qquad (4.10)$$

where w is the searcher's speed and the search commences at $\tau = 0$. We will use this expression as representative of the random search formulation for a fleeing-datum search problem.

In order to use Eq. (4.9), we must first understand the implications of using the definite-range law of detection. Strictly speaking, it results in an unbounded and discontinuous search density $\gamma(x,t)$, thereby violating assumptions (3.11) and (3.12). Although a formal argument employing the limit of a sequence of bounded and continuous functions could be made, a more direct way to justify the use of the definite-range concept comes from a physical interpretation of Eq. (4.8).

For fixed $x \in E^2$, $\theta \in [0,2\pi)$, and $t \in [0,\infty)$, Eq. (4.8) weighs $\exp\left[-\int_0^t \gamma[x - v_\theta(t-s), s]ds\right]$ by the value of the initial density at $x - v_\theta t$. For search path $z_1(s) \in E^2$, $s \in [0,\infty)$, $\gamma(y,s) = \gamma_1[y, z_1(s)]$ is infinite when $|y - z_1(s)| \le d$, and zero otherwise. The motion of the target implicit in the integral $\int_0^t \gamma_1[x - v_\theta(t-s),\ z_1(s)]ds$ is a straight-line path from $y_1 = x - v_\theta t$ at $s = 0$, to $y_2 = x$ at $s = t$. Therefore, if the target moving on this path ever comes within range d of the searcher at $z_1(s)$, then $\int_0^t \gamma_1[x - v_\theta(t-s),\ z_1(s)]ds = \infty$, and $p_\theta(x,t) = 0$; otherwise, $\int_0^t \gamma_1[x - v_\theta(t-s),\ z_1(s)]ds = 0$, and $p_\theta(x,t) = \rho_o(x - v_\theta t)$. These observations lead to a remarkably simple numerical procedure for evaluating Eq. (4.9) when a definite-range law of detection is assumed.

Consider the following inputs to a hypothetical submarine search problem:

1. The initial target density is given by

$$\rho_o(x) = \frac{1}{2\pi\sigma^2} \exp\left[-(x_1^2 + x_2^2)/2\sigma^2\right] \quad , \quad x = (x_1, x_2) \in E^2 \quad ,$$

where $\sigma = 10$ nmi.

2. The submarine moves as a fleeing datum with speed $u = 3$ kt

3. The search is undertaken at $T_1 = 3$ hr by a single searcher using a definite-range law sensor with $d = 2$ nmi.

4. The searcher flies at constant speed $w = 200$ kt along the path $z_1(\tau) = [z_{11}(\tau), z_{12}(\tau)] \in E^2$, $\tau = t - T_1$, where

$$z_{11}(\tau) = r(\tau) \cos\left\{\frac{\pi}{d}\,[D - r(\tau)]\right\} \quad ,$$

$$z_{12}(\tau) = r(\tau) \sin\left\{\frac{\pi}{d}\,[D - r(\tau)]\right\} \quad ,$$

$$r(\tau) = \left(D^2 - \frac{2w\,d}{\pi}\right)^{1/2} \quad ,$$

and $D = 32$ nmi. The search terminates at $T_2 = 7$ hr.

The search path defined in item 4 is an inward spiral starting at $D = 32$ nmi from the origin when $t = T_1 = 3$ hr, and ending at approximately 2 nmi from the origin when $t = T_2 = 7$ hr.

The probability of detection for this problem was calculated with the help of Eq. (4.9) and plotted in Fig. 4.1 under the heading "inward spiral search path." The calculation was repeated for a second search path $z_1^*(\tau) = z_1(T_2 - \tau)$, $\tau \in [0, T_2 - T_1)$, which is simply an unfolding of the first path. Figure 4.1 plots the result of the second calculation under the heading "outward spiral search path." The results of these two calculations are intuitive: The outward spiral path quickly results

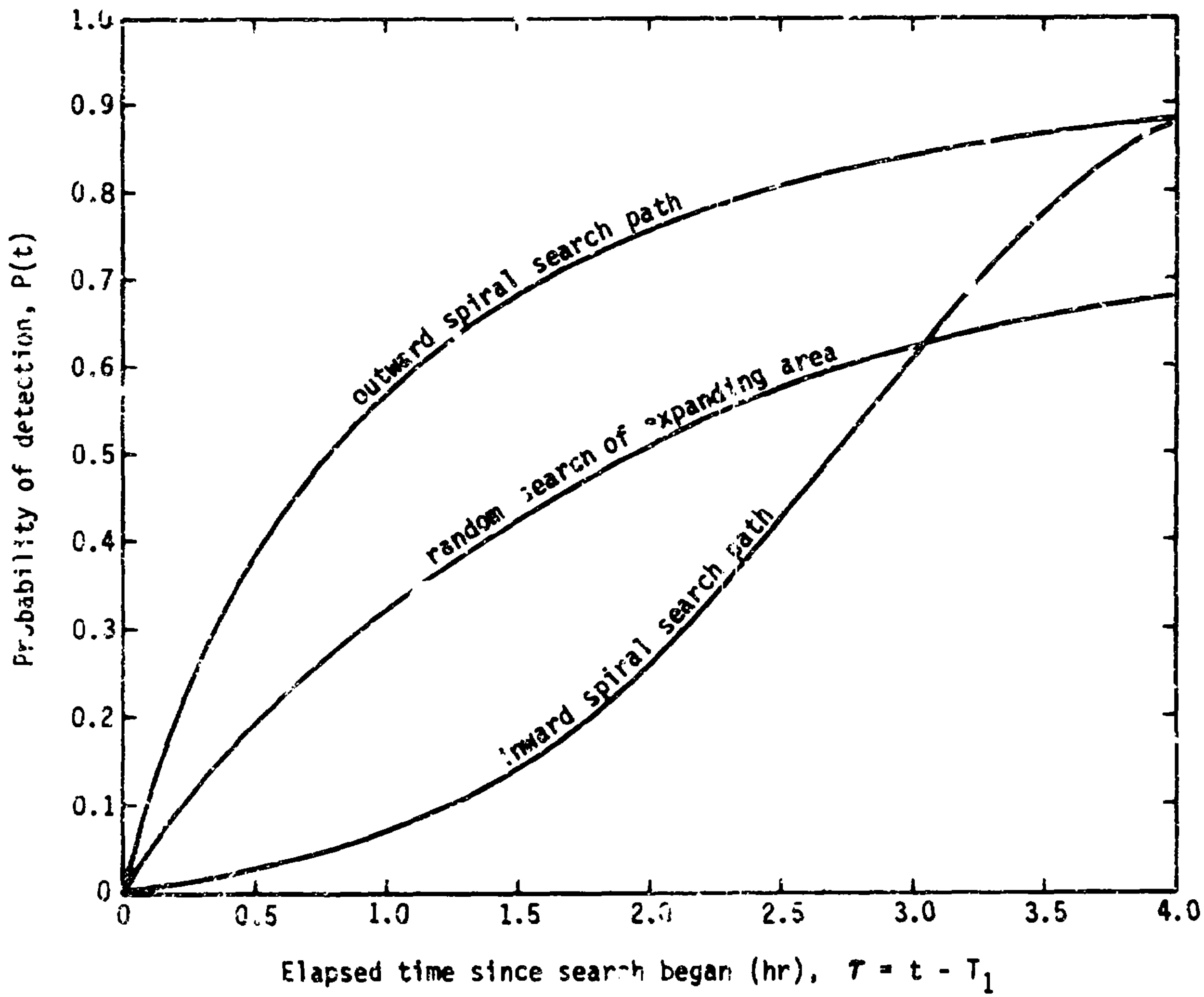


Fig. 4.1--Probability of detection for two search paths and a random search model

in a relatively high probability of detection, since the searcher is in a relatively "dense" target area for about the first two hours of search; the inward spiral path takes about two hours before rapidly increasing the probability of detection, since, after that time, the searcher finally begins to move through a relatively dense target area.

Figure 4.1 also graphs the probability of detection calculated using Eq. (4.10) for random search of an expanding area. The input values for this calculation are taken from the first two calculations as applicable, i.e., $u = 3$ kt, $d = 2$ nmi, and $w = 200$ kt. Since the search commences at $t = T_1 = 3$ hr, we chose $R_1 = 1.5\sigma + uT_1 = 24$ nmi. This choice is arbitrary, but the main conclusion to be drawn from Fig. 4.1 does not depend on it: the path-invariant random search formulation is not a good estimate of search system performance for known search tactics.

5.0 NUMERICAL OPTIMIZATION

The theory of optimal search for a stationary target has received considerable attention; the recent book by Stone [2] presents a unified treatment for much of that work. Saretsalo [7], working with Hellman's quation for detecting a Markovian target, has presented a necessary condition for the optimality of the search density $\gamma(x,t)$ in the sense of maximizing the probability of detection P(T) for fixed $T \in [0,\infty)$. Saretsalo's result is directly applicable to the search formulation of Sec. 3.0, but the question of sufficiency limits its utility.

The computational simplicity of Eq. (4.9) suggests the use of numerical techniques to optimize the search for a fleeing datum. Therefore, this section considers the numerical optimization of a stationary search (i.e., a search with density $\gamma(x,t) = \gamma(\cdot)$ for all $x \in \mathfrak{X}$ and $t \in [T_1, T_2)$) for a fleeing datum with a circular normal initial density. We constrain the search density $\gamma(x)$ to be the result of using n = 4, 8, or 12 definite-range-law sensors equally spaced on the circumference of a circle with radius R centered at the origin of E^2. R is then chosen to maximize $P(T_2)$, the probability of detection over the search interval $[T_1, T_2)$. The performance of the best circular pattern using eight sensors is compared to that of an optimal square pattern with nine sensors and shown to be superior. Finally, we determ ie the effect of using a fleeing datum circular pattern to find a target that in fact randomly chooses a new course every ΔT time unite. The section concludes with suggestions for broadening o r computational experience and extending the analytical development of the search formulation.

5.1 STATIONARY SEARCH FOR A FLEEING DATUM

This subsection performs a numerical optimization of a constrained stationary search for a fleeing datum. Initially, the target is assumed to be located according to the circular normal density

$$\rho_0(x) = \frac{1}{2\pi\sigma^2} \exp\left[-(x_1^2 + x_2^2)/2\sigma^2\right] \quad , \quad x = (x_1, x_2) \in E^2 \quad . \qquad (5.1)$$

Target motion for $t \geq 0$ is given by the fleeing-datum transition probability function of Eq. (4.6). We restrict our attention to the class of search densities defined by

$$\gamma(x,t) = \begin{cases} 0 & t \in [0, T_1) \\ \gamma(x) & t \in [T_1, T_2) \\ 0 & t \in [T_2, \infty) \end{cases} \quad . \qquad (5.2)$$

A search using a member of this class of densities is referred to as a *stationary search* with time late T_1 and duration $T_2 - T_1$. The following numerical optimization seeks to maximize $P(T_2)$, the probability of detection during the interval $[T_1, T_2)$, over a specific family of sensor configurations that satisfy Eq. (5.2).

The solution of Eq. (4.7), the fleeing-datum linear search equation conditioned on target heading $\theta \in [0, 2\pi)$, when $\gamma(x,t)$ is given by Eq. (5.2), is

$$p_\theta(x,t) = \begin{cases} \rho_0(x - v_\theta t) & t \in [0, T_1) \\ \rho_0(x - v_\theta t) \exp\left[-\int_{T_1}^{t} \gamma[x - v_\theta(t-s)]\, ds\right] & t \in [T_1, T_2) \end{cases} \quad ,$$

where $v_\theta = u(\cos\theta, \sin\theta)$ is the target velocity vector with known speed u. The probability of detection in $[T_1, T_2)$ is thus given by

$$P(T_2) = 1 - \frac{1}{2\pi} \int_x \int_{\theta=0}^{2\pi} \rho_0(x - v_\theta T_2) \exp\left[-\int_{T_1}^{T_2} \gamma[x - v_\theta(T_2 - s)]ds\right] d\theta dx \quad , \tag{5.3}$$

where, for the numerical work below, we assume that $\rho_0(x)$ is circular normal.

The search is assumed to involve the use of n definite-range-law sensors with detection range d. The sensors are assumed to be independent in the sense of Eq. (4.1). (Subsection 4.4 discusses the computational implications of assuming definite-range-law sensors.) We seek the optimum placement of the n sensors such that the probability of detection calculated from Eq. (5.3) is maximized. However, we do not attempt a global optimization, but instead constrain the sensors to lie equally spaced on the circumference of a circle centered about $x = 0$, the mean of $\rho_0(x)$--see Fig. 5.1 for n = 8 sensors--and then calculate the radius R of the circle that maximizes $P(T_2)$ As long as the circular detection areas of the individual sensors do not overlap (as will be the case for our analysis), the circular pattern is reasonable in view of the radial symmetry of both the initial density and the fleeing datum motion.

Placing the optimization in the context of a search for a fleeing submarine by use of moored acoustic sensors (i.e., sonobuoys), the following values were chosen for the input parameters: n = 4, 8, or 12 sonobuoys; u = 6 or 12 kt; σ = 20 or 40 nmi; and d = 4 or 10 nmi.

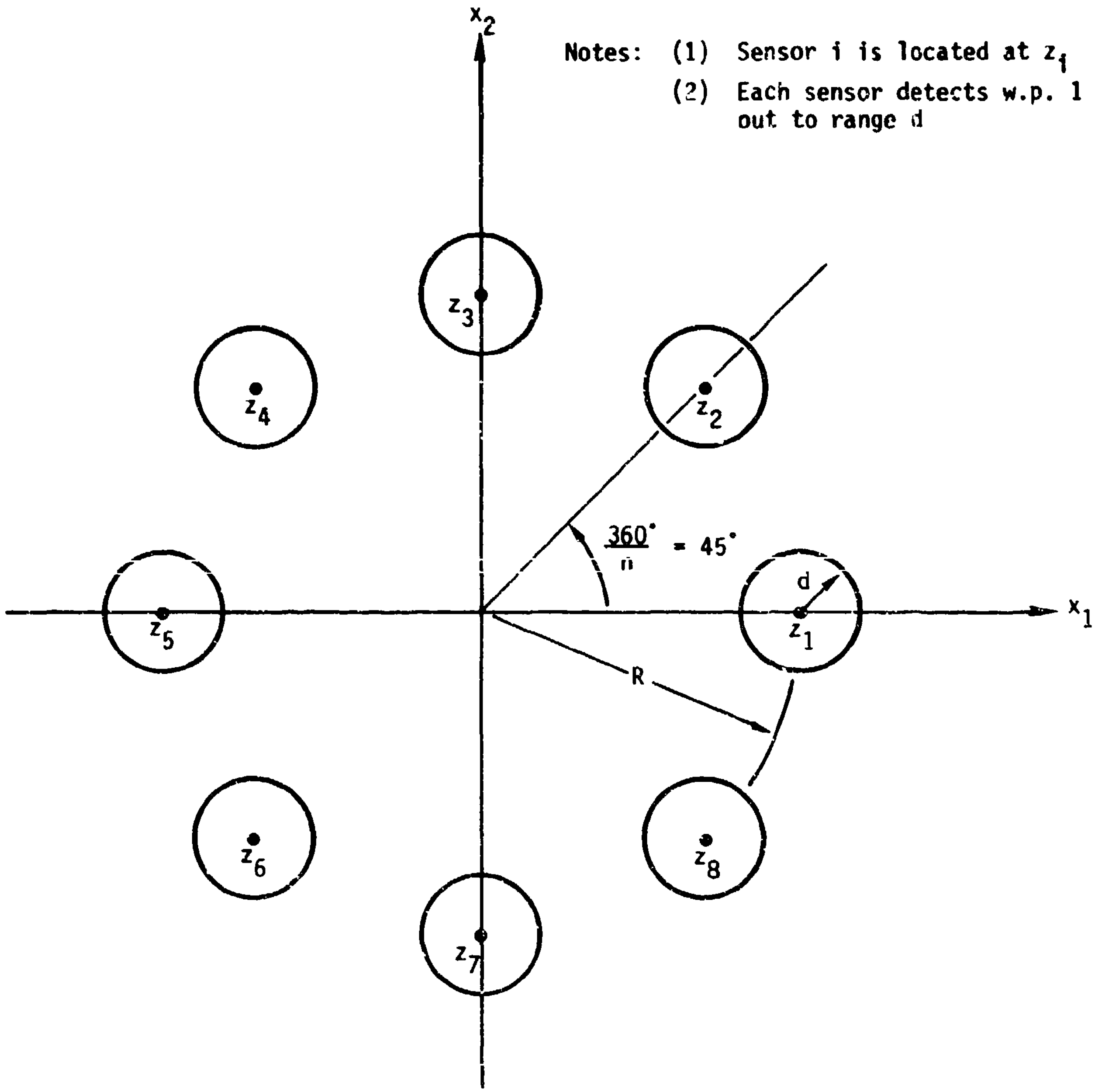


Fig. 5.1--Circular pattern of n = 8 definite-range-law sensors used to detect a fleeing datum

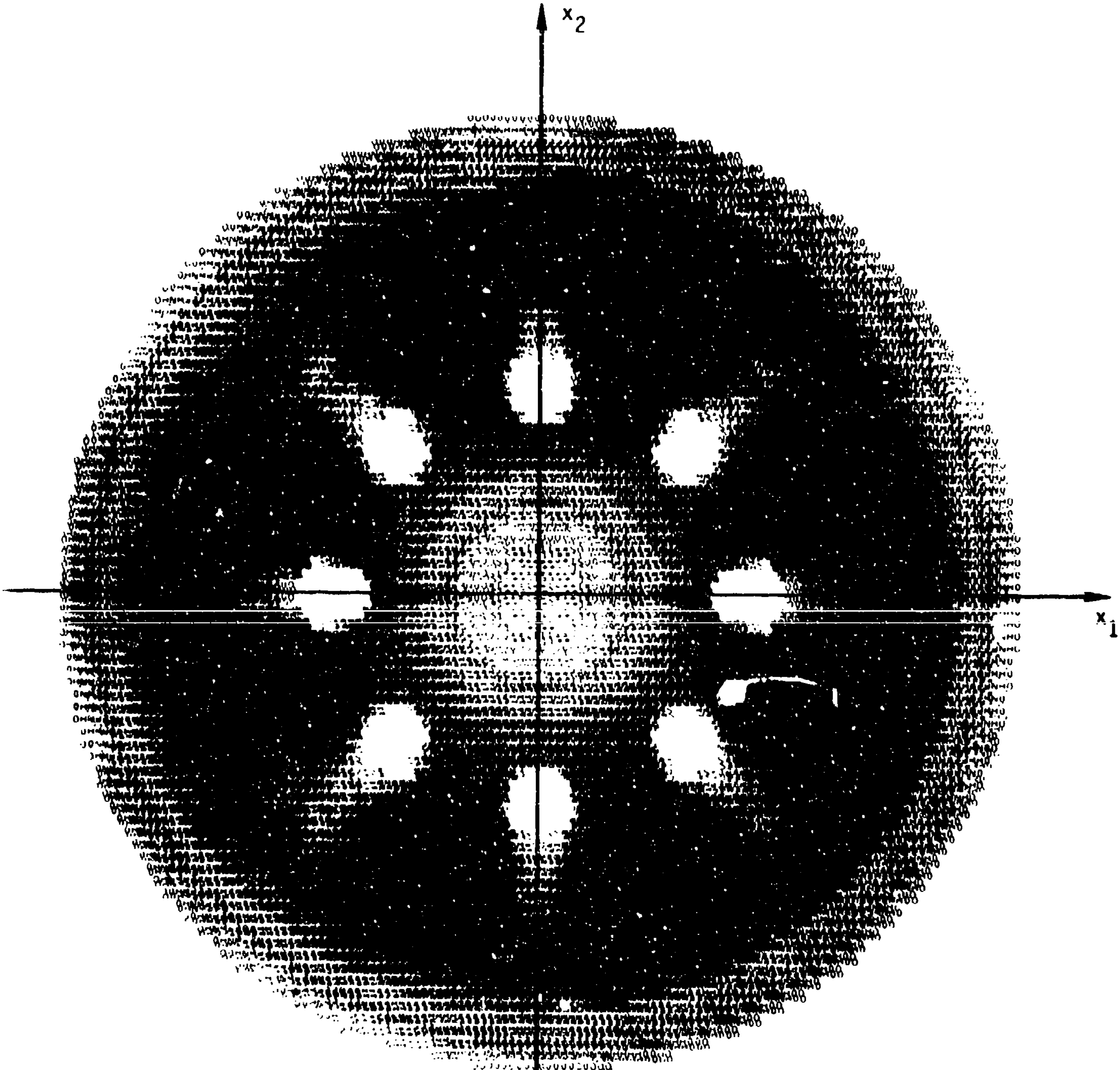
The time late T_1 and search duration $T_2 - T_1$ were fixed at 4 and 6 hr, respectively. The table below lists the radius of the optimum circular sonobuoy pattern, and resulting probability of detection, for each combination of the parameters n, u, σ, and d. Figure 5.2 plots $\rho(x,t)$, $t = 7$ hr, the conditional density midway into the search for case 14 in the table. The darker an area, the more likely the target is within that area, assuming an unsuccessful search. To show the sensitivity of $P(T_2)$, the detection probability, to changes in R, the radius of circular sonobuoy pattern, Fig. 5.3 plots $P(T_2)$ against R for cases 10, 12, 14, and 16.

Figure 5.4 plots $P^*(T_2)$, the probability of detection for the optimum circular sonobuoy pattern, against n, the number of sonobuoys deployed. Each of the eight curves corresponds to a unique combination of u, σ, and d. The plot could be used, say, to conclude that a search for a fleeing submarine using only four sonobuoys, each with a 10 nmi detection range, is about as effective as a search using 12 sonobuoys, each having a 4 nmi detection range.

To demonstrate the effectiveness of the circular sonobuoy pattern, we calculated the probability of detection for a square pattern, using nine sonobuoys as shown in Fig. 5.5. We chose the distance between sonobuoys L to maximize $P(T_2)$ for speed $u = 12$ kt, detection range $d = 10$ nmi, and initial standard deviation $\sigma = 20$ nmi--as in all cases using the circular patterns $T_1 = 4$ hr and $T_2 = 10$ hr. The maximum $P^*(T_2) = 0.35$ was achieved for $L^* = 53$ nmi; Fig. 5.6 plots $\rho(x,t)$, $t = 7$ hr for $L = L^*$. Thus, the optimal square pattern with *nine* sonobuoys was slightly less effective than the optimal circular pattern using *eight* sonobuoys (case 8).

OPTIMUM CIRCULAR SONOBUOY PATTERNS

Case	Number of Sonobuoys (n)	Target Speed (u kt)	Initial Standard Deviation (σ nmi)	Detection Range (d nmi)	Optimum Pattern Radius (R^* nmi)	Maximum Probability of Detection ($P^*(T_2)$)
1	4	6	20	4	25	0.11
2			20	10	26	0.32
3			40	4	24	0.07
4			40	10	29	0.21
5		12	20	4	68	0.07
6			20	10	62	0.18
7			40	4	59	0.05
8			40	10	58	0.14
9	8	6	20	4	30	0.21
10			20	10	32	0.58
11			40	4	28	0.13
12			40	10	35	0.35
13		12	20	4	67	0.13
14			20	10	62	0.36
15			40	4	63	0.10
16			40	10	58	0.28
17	12	6	20	4	32	0.31
18			20	10	40	0.75
19			40	4	29	0.19
20			40	10	44	0.44
21		12	20	4	67	0.20
22			20	10	62	0.54
23			40	4	65	0.15
24			40	10	65	0.40



Notes: Target speed: $u = 12$ kt

Initial standard deviation: $\sigma = 20$ nmi

Sonobuoy detection range: $d = 10$ nmi

Pattern radius: $R = 32$ nmi

Time late: $T_1 = 4$ hr

Density plot: $\rho(x,t)$, $t = 7$ hr

Fig. 5.2--Target density $\rho(x,t)$ for an unsuccessful search using a circular pattern with $n = 8$ sonobuoys

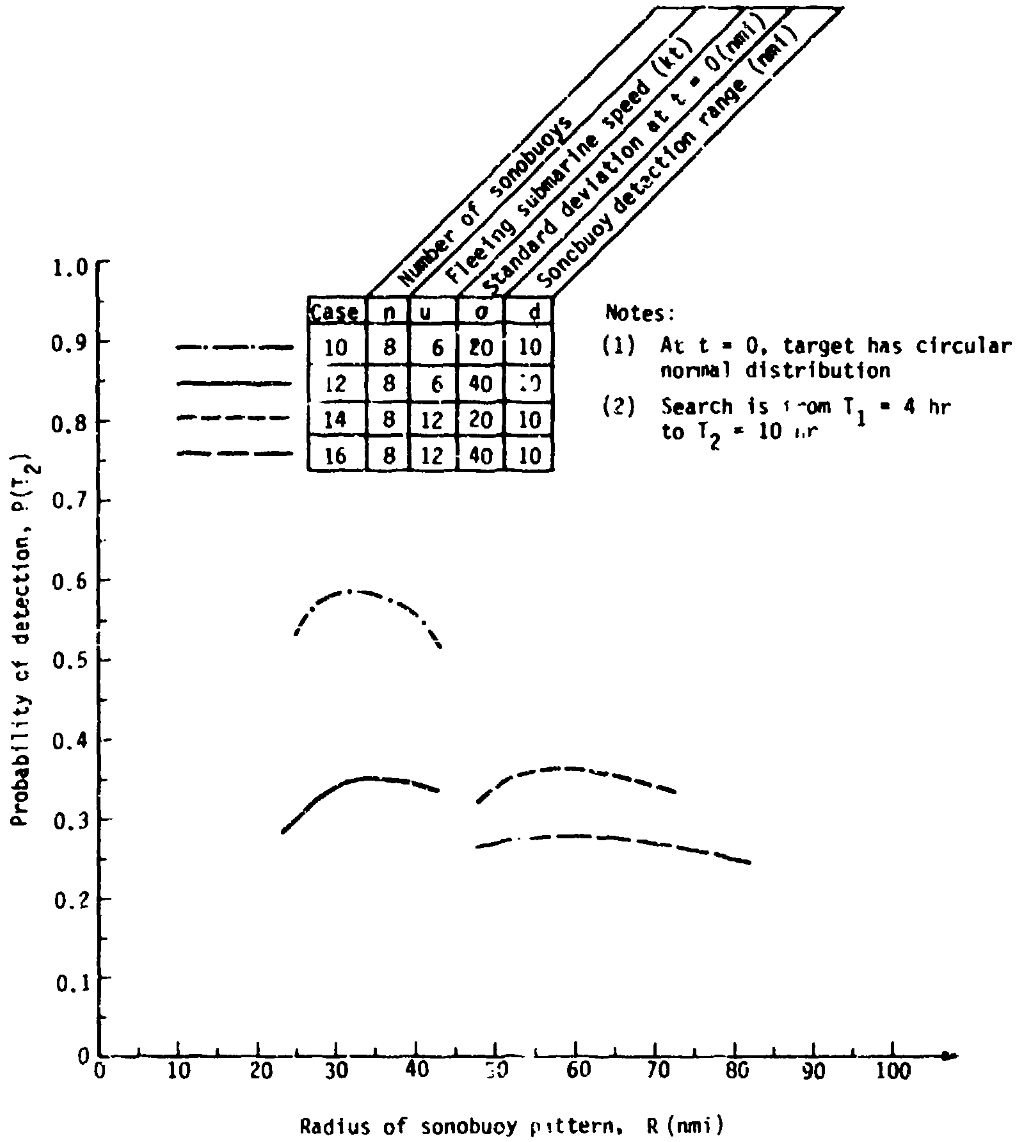


Fig. 5.3--Probability of detection versus radius for representative circular sonobuoy patterns

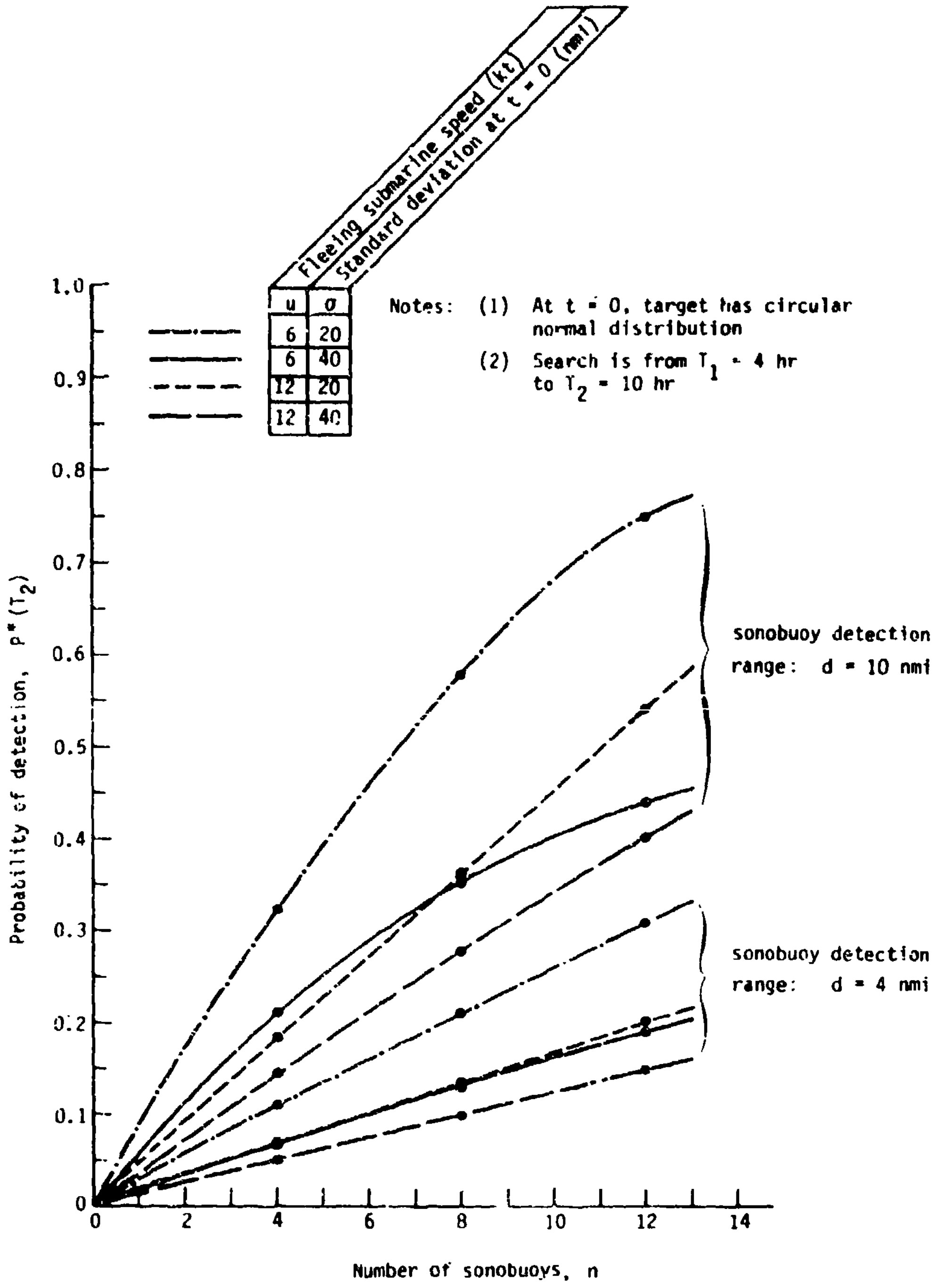


Fig. 5.4--Probability of detection for optimum circular sonobuoy patterns used to detect a fleeing submarine

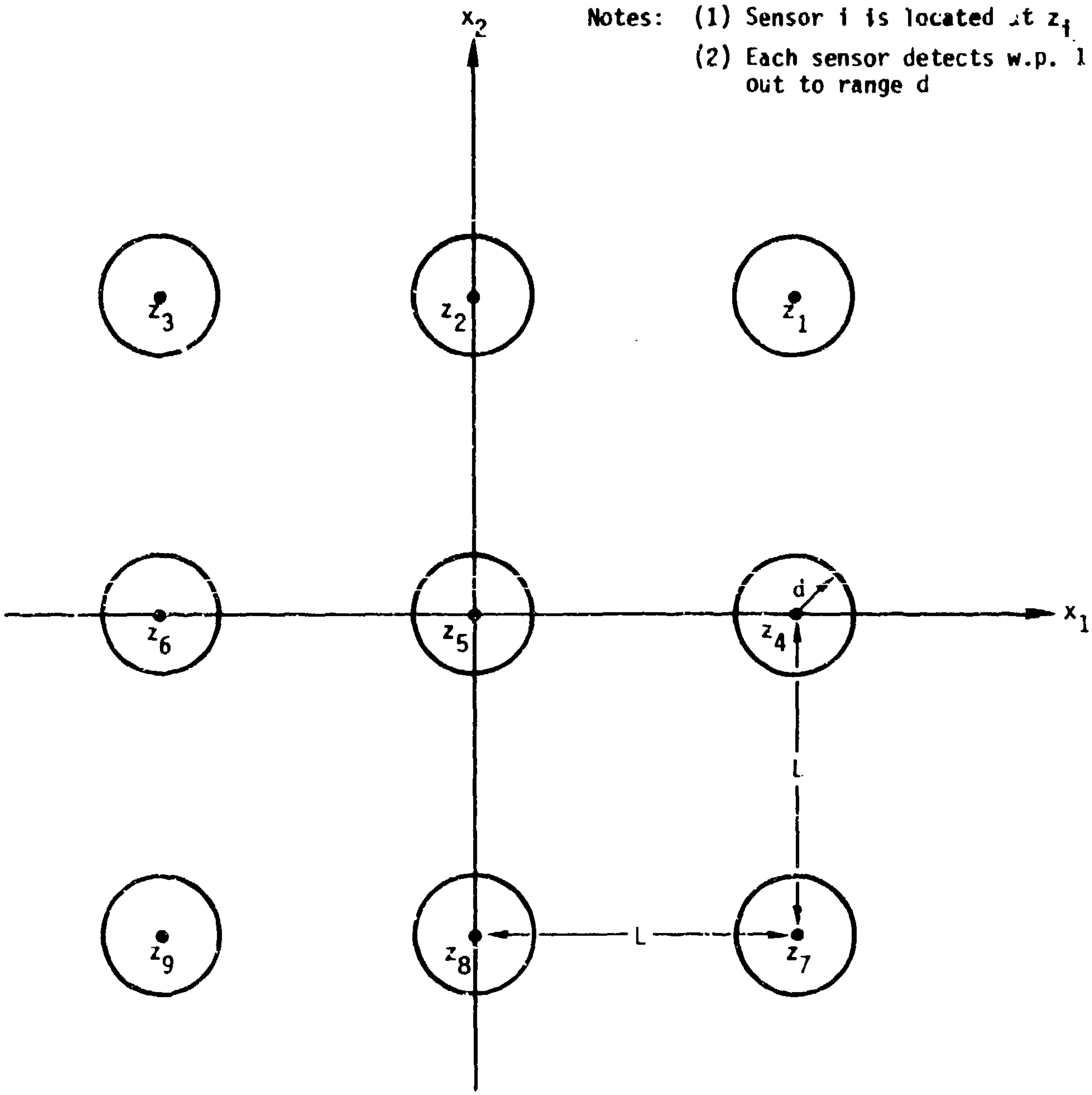
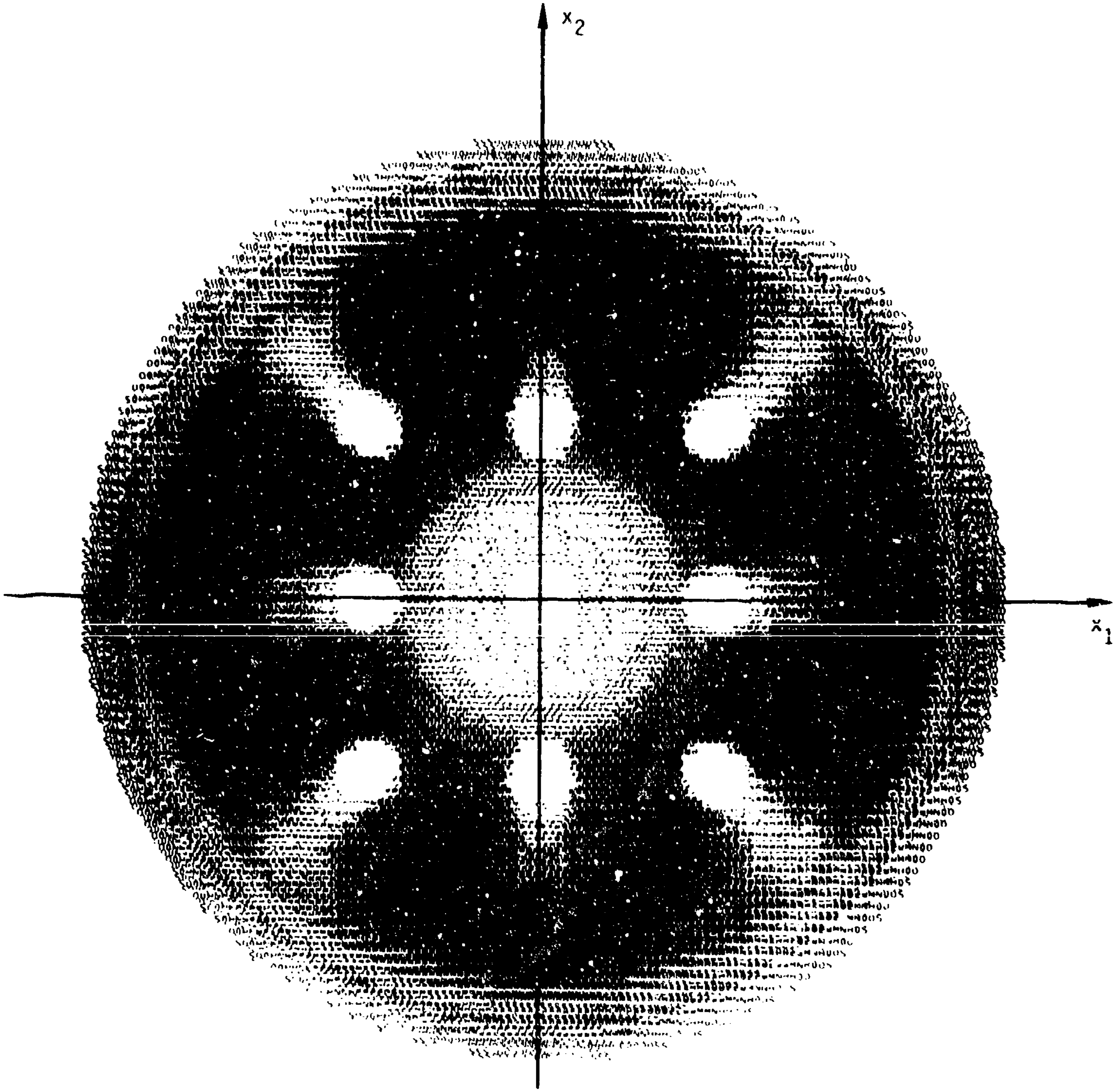


Fig. 5.5--Square pattern of n = 9 definite-range-law sonobuoys used to detect a fleeing submarine



Notes: Target speed: u = 12 kt

Initial standard deviation: σ = 20 nmi

Sonobuoy detection range: d = 10 nmi

Distance between sonobuoys: L = 53 nmi

Time late: T_1 = 4 hr

Density plot: $\rho(x,t)$, t = 7 hr

Fig. 5.6--Target density $\rho(x,t)$ for an unsuccessful search using a square pattern with n = 9 sonobuoys

5.2 EFFECT OF TARGET COURSE CHANGES ON OPTIMIZATION

We now consider the effect of deploying an optimal circular sonobuoy pattern to detect a target that is assumed to move as a fleeing datum when in fact it randomly chooses a new course every ΔT time units. Recall that a fleeing datum moves on a straight-line path after a single random heading selection at $t = 0$. As in subsection 5.1, we assume that the target distribution at $t = 0$ is circular normal, and that the search commences at time late $T_1 > 0$. The target is assumed to move as a fleeing datum until $t = T_1$; however, at $t = T_1$, the target selects a new course from a uniform distribution on $[0,2\pi)$. Furthermore, the target continues to select a new course every ΔT time units corresponding to, say, a target that becomes aware of the sensor deployment at $t = T_1$ and subsequently attempts to avoid detection.

At $t = T_1$, the target location density is the solution of the search-free equation (3.32) for a fleeing datum. Specifically, for $x = (x_1, x_2) \in E^2$,

$$\rho(x,T_1) = \frac{1}{2\pi\sigma^2} \exp[-(r^2 + u^2 T_1^2)/2\sigma^2] I_o (ruT_1/\sigma^2) \quad , \qquad (5.4)$$

where $r^2 = x_1^2 + x_2^2$, and I_o denotes the ordinary Bessel function of order zero with an imaginary argument. Since the search commences at the time of the first new course selection, Eq. (5.4) can be interpreted as the initial density of a fleeing-datum search problem with time late zero and search duration ΔT. The joint density at $t = T_1 + \Delta T$ can then be computed using Eq. (4.8) as

$$p(x,T_1+\Delta T) = \frac{1}{2\pi}\int_0^{2\pi} \rho(x - v_{\theta_1}\Delta T, T_1)\ \exp\left[-\int_{T_1}^{T_1+\Delta T} \gamma[x - v_{\theta_1}(T_1+\Delta T - s)]ds\right]d\theta_1 \quad .$$

The probability of no detection in the interval $[T_1, T_1+\Delta T)$ is simply

$$Q_{\Delta T}(1) = \int p(x,T_1+\Delta T)dx \quad .$$

From Eq. (3.28), the conditional density at $t = T_1 + \Delta T$ is

$$\rho(x,T_1+\Delta T) = Q_{\Delta T}^{-1}(1)\ p(x,T_1+\Delta T) \quad .$$

Now using $\rho(x,T_1+\Delta T)$ as the initial density for a fleeing datum search problem in the interval $[T_1+\Delta T, T_1+2\Delta T)$, we can compute $Q_{\Delta T}(2)$, the probability of no detection in that interval, and so on. This procedure obtains the probability of detection in the interval $[0,T_2)$ given by

$$P_m(T_2) = 1 - \prod_{k=1}^{m} Q_{\Delta T}(k) \quad , \qquad (5.5)$$

where ΔT is conveniently chosen such that $m\Delta T = T_2 - T_1$.

For case 10 in the table, the radius of the optimum circular pattern of eight sonobuoys is $R^* = 32$ nmi. Holding all other inputs the same as in case 10 (i.e., submarine speed $u = 6$ kt, initial standard deviation $\sigma = 20$ nmi, and sonobuoy detection range $d = 10$ nmi), we use this circular pattern to compute $P_m(T_2)$ from Eq. (5.5) as a function of m, the number of course changes in the 6 hr interval $[T_1,T_2)$. The results are graphed in Fig. 5.7. The value of $P_m(T_2)$ at $m = 0$ is, of course, the

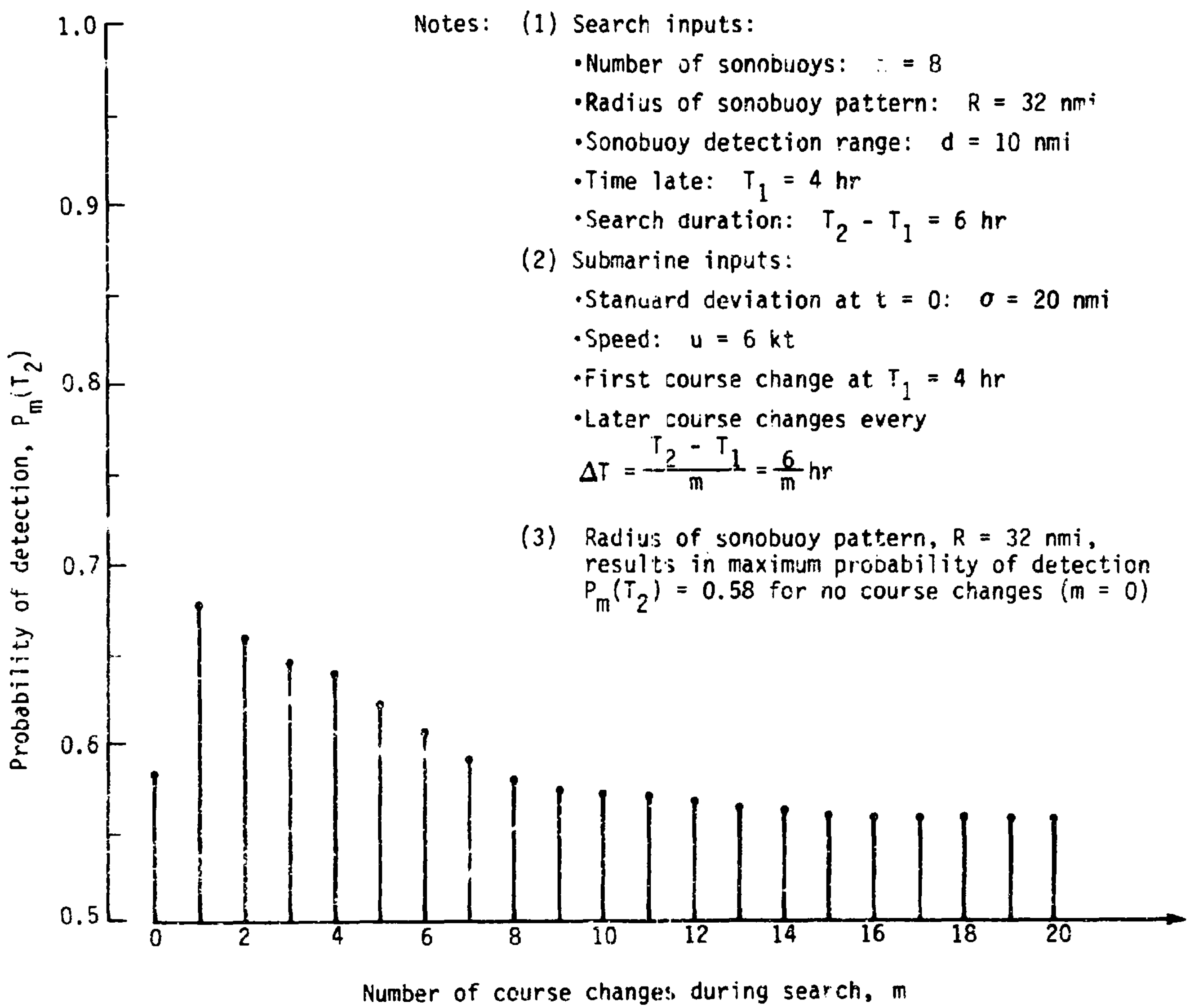


Fig. 5.7—Effect of submarine course changes on probability of detection using a sonobuoy pattern optimized for no course changes

probability of detection for the optimum circular sonobuoy pattern when the submarine moves as a simple fleeing datum. It is of some interest to note that $P_m(T_2) > P_0(T_2) = 0.58$ for $m = 1,2,\ldots,7$. A submarine attempting to avoid detection by a circular sonobuoy pattern optimized for a fleeing datum would thus actually *decrease* its probability of escape by making a few additional course selections. Furthermore, Fig. 5.7 suggests that an optimum fleeing-datum search pattern is not significantly compromised by any number of course changes in the interval $[T_1,T_2)$ (i.e., $|P_m(T_2) - P_0(T_2)| < 0.03$ for all $m > 7$). Although our example is somewhat idealized, these results may warrant further investigation with more realistic inputs.

As a final calculation for the inputs of case 10 and m = 6 course changes, Fig. 5.8 plots $P_m(T_2)$ from Eq. (5.5) against R, the radius of a circular sonobuoy pattern. The optimum circular pattern of eight sonobuoys is shown to have a radius of 28 nmi when the submarine is assumed to select a new bearing every hour. This compares with an optimum radius of 32 nmi for the simple fleeing datum motion.

5.3 EXTENSIONS

The purpose of this report is to 1) render the search formulation accessible to operations analysts without sacrificing mathematical rigor; 2) narrow the gap between certain analytical assumptions and the behavior of actual targets and existing search devices; and 3) illustrate the utility of the formulation with simple numerical optimization procedures. The developments herein should be regarded as an introduction to this work. Some suggestions for further research are given below.

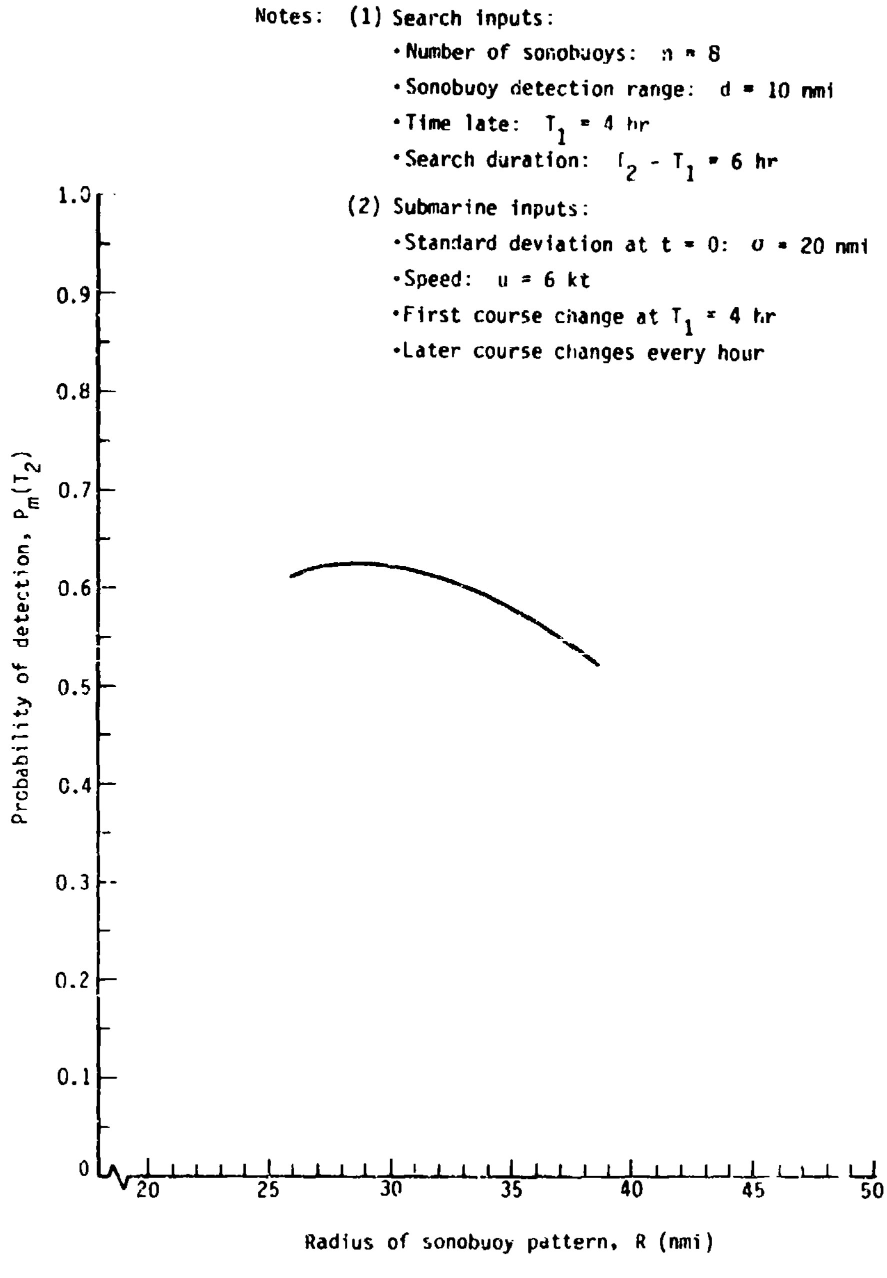


Fig. 5.8--Effect of sonobuoy pattern radius on probability of detection for a submarine that changes course every hour

The numerical analysis of this section would be more relevant to an actual search if a modest effort were made to improve the characterization of target and searcher behavior. For example, by obtaining analytical solutions of Eq. (3.30) for 1) the class of conditionally deterministic target motion or 2) more general diffusion motions, we could perform the numerical work of this section for a much broader and realistic set of targets.

With regard to the search process, it would be desirable to solve numerical problems for search devices with more sophisticated laws of detection than the definite-range concept. Perhaps an integration-type sensor (see subsection 4.2) subject to "convergence zone" phenomena [8] would be a more realistic device to study. More generally, future numerical work should include the constrained optimization of a moving searcher problem--i.e., from a given family of paths that a moving searcher might follow, find the path that results in the highest probability of detection in a given search problem.

On a more fundamental level, work should be done on the problems of evasive targets, false targets and false contacts, multiple targets, and search devices that detect as a function of the relative orientation of target and searcher. Finally, for a general Markovian search, we have yet to solve the problems of obtaining sufficient conditions for an optimal search density and how to obtain such a density having found those conditions.

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